#3.3.7

How to find a parametrization for a surface?

When we use parametrization to represent a surface, we express the coordinates $(x, y, \text{and } z)$ of all points on the surface as functions of two parameters (say, u and v). In particular, if we can express one of the coordinate variables $(x, y, \text{and } z)$ in terms of the other two, then we can use the two parameters to represent those two coordinate variables and use the relationship among the three variables to find a formula for the remaining variable in terms of the parameters.

Case 1 (Simple Case): If the surface is part of the graph of a function $z = z(x, y)$, then one easy way (that always works) to parametrize such surface is by using $x = u$, $y = v$, and $z = z(x, y)$; for instance, if we were to parametrize the triangular surface ABC in the example on page 95, since it is a portion of the graph of the plane with equation $3x + 2y + z = 6$ (which is equivalent to the equation $z = 6 - 3x - 2y$), then one way to parametrize it is by using $x = u$, $y = v$, $z = 6 - 3x - 2y$.
Note: We have this simple case 1 if and only if the surface in question passes the vertical line test (that is, each vertical line only passes the surface once; in other words, each pair of values (x, y) corresponds to only one z-value).

Case 2 (Tricky Case): The surface fails the vertical line test, such as the surface OAC in the example on page 95. [Any vertical line intersects the surface OAC at more than one point – actually at infinitely many points.]
There is no fool-proof method to parametrize surfaces in this case 2 – we need to analyze the particular surface in hand to find a parametrization.
We observe that OAC is part of the xz-plane and hence the y-coordinate of all points in this surface is always zero (that is, $y = 0$).
Then the real question now is how we can express the x-coordinate and z-coordinate of the points in OAC as functions of two parameters u and v.
In the solution in the package, we use one parameter (namely u) for the x-coordinate (hence $x = u$), as for the other parameter v, we use it as the parameter in the parametric equation for the vertical line with the generic $x(= u)$ – more precisely, we use v as the parameter for the vertical line segment joining the point $(u, 0, 0)$ (on OA) to the point $(u, 0, 6 - 3u)$ (on AC).
From Linear Algebra Math 251 last term, we learned that the parametrization is given by Initial + Parameter * (Terminal – Initial), hence

$$[u, 0, 0] + v([u, 0, 6 - 3u] - [u, 0, 0])$$

which simplifies to $[u, 0, (6 - 3u)v]$.

For problem #3.3.7, the boundary surface $ABED$ of T fails the vertical line test.
We observe that it is part of the vertical plane with equation $y = 2 - x$ and hence the y-coordinate of all points in this surface is always related to the x-coordinate via this equation. If we use one parameter (say u) for the x -coordinate (hence $x = u$), then the matching formula for the y-coordinate would be $y = 2 - u$.
The remaining question now is how we can express the z-coordinate of the points On $ABED$ as functions of two parameters u and v.
One way is to look at this surface as a collection of vertical line segments.
For the generic one with $x = u$ and $y = 2 - u$, the endpoints of this vertical line segment have coordinates $(u, 2 - u, 0)$ and $(u, 2 - u, 9 - u^2)$. We can represent all points on this line segment by using the other parameter v as the parameter in the parametric equation for this line – more precisely, from Linear Algebra Math 251 last term, we learned that the parametrization is given by Initial + Parameter * (Terminal – Initial), hence

$$[u, 2 - u, 0] + v([u, 2 - u, 9 - u^2] - [u, 2 - u, 0])$$

which simplifies to $[u, 2 - u, (9 - u^2)v]$. (Fill in the details.)
For this parametrization representing the points on the surface $ABED$, since those x -coordinates lie between 0 and 2, we have $0 \leq u(= x) \leq 2$;
as we use v for the parameter (t in Linear Algebra Math 251) for the parametrization of the line joining initial point and terminal point, we have $0 \leq v(= t) \leq 1$
[$t = 0$ corresponds to the initial point:

Initial + (0) * (Terminal – Initial) = Initial

and $t = 1$ corresponds to the terminal point:

Initial + (1) * (Terminal – Initial) = ~~Initial~~ + (Terminal – ~~Initial~~) = Terminal].