Self-Test on Mathematics Readiness
(for students entering the Engineering Bridge Programs)

- Use only scientific (non-graphing, non-programmable) calculator.
- Simplify all your answers and give them in exact form when possible.
- Each question is labeled "must know" or "nice to know".

1. (Must Know) Solve the inequality $\left|\frac{x+5}{4}\right| \le 1$ [Answer: $-9 \le x \le -1$ or $[-9,-1]$]

Solution:

$$-1 \le \frac{x+5}{4} \le 1 \Longrightarrow 4(-1) \le x+5 \le 4(1) \Rightarrow -4 \le x+5 \le 4$$

$$\Rightarrow -4-5 \le x \le 4-5 \Rightarrow \mathbf{-9 \le x \le -1}$$

2. (Must Know) Given n is a positive integer, evaluate and simplify $\frac{(n+1)^2 2^{n+2}}{3^{n+1}} \div \frac{n^2 2^{n+1}}{3^n}$.

[Answer: $\frac{2(n+1)^2}{3n^2}$]

Solution:

$$\frac{(n+1)^2 2^{n+2}}{3^{n+1}} \div \frac{n^2 2^{n+1}}{3^n} = \frac{(n+1)^2 2^{n+2}}{3^{n+1}} \times \frac{3^n}{n^2 2^{n+1}} = \frac{2^{n+2}}{2^{n+1}} \times \frac{3^n}{3^{n+1}} \times \frac{(n+1)^2}{n^2}$$

$$= 2^{(n+2)-(n+1)} \times 3^{n-(n+1)} \times \frac{(n+1)^2}{n^2} = 2 \times 3^{-1} \times \frac{(n+1)^2}{n^2} = \mathbf{\frac{2(n+1)^2}{3n^2}}$$

3. (Must Know) Perform the long division $\frac{2x^3+7x^2-5}{x+3}$ [Answer: $\frac{2x^3+7x^2-5}{x+3} = 2x^2 + x - 3 + \frac{4}{x+3}$]

Solution:

$$\begin{array}{rr|rrrrl}
 & & & 2x^2 & +x & -3 & = \text{Quotient} \\
\cline{3-6}
x+3 & & 2x^3 & +7x^2 & +0x & -5 & \\
 & -) & 2x^3 & +6x^2 & & & \\
\cline{3-6}
 & & & x^2 & +0x & & \\
 & & -) & x^2 & +3x & & \\
\cline{4-6}
 & & & & -3x & -5 & \\
 & & & -) & -3x & -9 & \\
\cline{5-6}
 & & & & & 4 & = \text{Remainder}
\end{array}$$

Hence $\frac{2x^3+7x^2-5}{x+3} = \mathbf{2x^2 + x - 3 + \frac{4}{x+3}}$.

4. (Must Know) Factor completely.

(a) $x - 64x^4$ [Answer: $x(1-4x)(1+4x+16x^2)$]

Solution:

$x - 64x^4 = x[1 - 64x^3] = x[1^3 - (4x)^3] = x(1-4x)[1^2 + (1)(4x) + (4x)^2]$

$= \mathbf{x(1-4x)(1+4x+16x^2)}$

(b) $12u^3 + 10u^2 - 8u$ [Answer: $2u(2u-1)(3u+4)$]

Solution:

$12u^3 + 10u^2 - 8u = 2u[6u^2 + 5u - 4] = 2u[6u^2 - 3u + 8u - 4]$

$= 2u[(6u^2 - 3u) + (8u - 4)] = 2u[3u(2u-1) + 4(2u-1)]$

$= \mathbf{2u(2u-1)(3u+4)}$

5. (Nice To Know) Find the modulus and the principal argument of $-3 + 4i$.

[Answer: $|-3+4i| = 5, arg(-3+4i) = \pi - \tan^{-1}\left(\frac{4}{3}\right) \approx 2.2143$ (rad)]

Solution:

Modulus $r = |-3+4i| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = \mathbf{5}$

$\begin{cases} r\cos\theta = -3 \\ r\sin\theta = 4 \end{cases} \Rightarrow \begin{cases} \cos\theta = -3/5 \\ \sin\theta = 4/5 \end{cases}$

$\Rightarrow \begin{cases} \tan\theta = \sin\theta/\cos\theta = -4/3 \\ \text{From the CAST daigram, the principal argument } \theta \text{ terminates in quadrant II} \end{cases}$

Hence the reference angle φ for θ is φ = arctan(|–4/3|) = arctan(4/3) and

the principal argument θ = π – φ = **π – arctan(4/3) ≈ 202143 (rad)**

6. (Nice To Know) Simplify $\frac{2-5i}{1-6i}$ and write your answer in the rectangular form $a + bi$.

[Answer: $\frac{32}{37} + \frac{7}{37}i$]

Solution:

$$\frac{2-5i}{1-6i} = \frac{(2-5i)(1+6i)}{(1-6i)(1+6i)} = \frac{2+12i-5i-30i^2}{1+6i-6i-36i^2}$$

$$= \frac{2 + 12i - 5i + 30}{1 + 36} = \mathbf{\frac{32}{37} + \frac{7}{37}i}$$

7. (Nice To Know) Solve the equation.

(a) $3w^2 + 2w = 7$ [Answer: $w = \frac{-1-\sqrt{22}}{3}, \frac{-1+\sqrt{22}}{3} \approx -1.897, 1.230$]

Solution:

(Rewrite the quadratic equation in standard form) $3w^2 + 2w - 7 = 0$

(Since the determinant $D = 2^2 - 4(3)(-7) = 88$ is positive but not a perfect square, the roots are real and irrational ; we cannot factor the quadratic expression and we use the quadratic formula)

$$w = \frac{-2 \pm \sqrt{2^2 - 4(3)(-7)}}{2(3)} = \frac{-2 \pm \sqrt{88}}{6} = \frac{2(-1 \pm \sqrt{22})}{6}$$

$$= \mathbf{\frac{-1 \pm \sqrt{22}}{3} \approx -1.897, 1.230}$$

(b) $x^2 + 5x + 8 = 0$ [Answer: $x = -\frac{5}{2} - \frac{\sqrt{7}}{2}i, -\frac{5}{2} + \frac{\sqrt{7}}{2}i$]

Solution:

(Since the determinant $D = 5^2 - 4(1)(8) = -7$ is negative, the roots are complex nos.; we cannot factor the quadratic expression and we use the quadratic formula)

$$w = \frac{-5 \pm \sqrt{5^2 - 4(1)(8)}}{2(1)} = \frac{-5 \pm \sqrt{-7}}{2} = \frac{-5 \pm \sqrt{7}i}{2} = \mathbf{\frac{-5}{2} \pm \frac{\sqrt{7}}{2}i}$$

(c) $x^4 + 4x^2 - 45 = 0.$ [Answer: $x = -\sqrt{5}, \sqrt{5}, -3i, 3i$]

Solution:

(Even though this is not a quadratic equation – the degree of the polynomial in this equation is 4 and not 2, this is a quadratic type equation and we can rewrite it into a quadratic equation by using the substitution $y = x^2$)

$(x^2)^2 + 4(x^2) - 45 = 0 \Rightarrow y^2 + 4y - 45 = 0 \quad (y = x^2)$

(Since the determinant $D = 4^2 - 4(1)(-45) = 196 = 14^2$ is positive and a perfect square, the roots for y are real rational numbers; we can solve the quadratic equation by factoring the quadratic expression)

$y^2 + 4y - 45 = 0 \Rightarrow (y - 5)(y + 9) = 0 \Rightarrow y = 5, -9$

(We complete the solution by solving for x using the values we found for y)

$$\begin{cases} x^2 = 5 \Rightarrow x = \pm\sqrt{5} \\ x^2 = -9 \Rightarrow x = \pm\sqrt{-9} = \pm 3i \end{cases} \Rightarrow \boldsymbol{x = -\sqrt{5}, \sqrt{5}, -3i, 3i}$$

8. (Must Know) Simplify $\frac{x}{x^2+11x+30} - \frac{5}{x^2+9x+20}$. [Answer: $\frac{x-6}{(x+6)(x+4)}$]

Solution:

$$\frac{x}{x^2+11x+30} - \frac{5}{x^2+9x+20} = \frac{x}{(x+6)(x+5)} - \frac{5}{(x+5)(x+4)}$$

$$= \frac{x(x+4)}{(x+6)(x+5)(x+4)} - \frac{5(x+6)}{(x+6)(x+5)(x+4)}$$

$$= \frac{(x^2+4x)-(5x+30)}{(x+6)(x+5)(x+4)} = \frac{x^2-x-30}{(x+6)(x+5)(x+4)}$$

$$= \frac{(x-6)\cancel{(x+5)}}{(x+6)\cancel{(x+5)}(x+4)} = \boldsymbol{\frac{x-6}{(x+6)(x+4)}}$$

9. (Must Know) Solve the equation $\frac{x^2+2x}{x-2} = \frac{8}{x-2}$. [Answer: $x = -4$]

Solution:

(Clear the denominators by multiplying both sides of the equation with $x - 2$, and rewrite the resulting quadratic equation in standard form)

$$x^2 + 2x = 8 \Rightarrow x^2 + 2x - 8 = 0$$

(Since the determinant $D = 2^2 - 4(1)(-8) = 36 = 6^2$ is positive and a perfect square, the roots for y are real rational numbers; we can solve the quadratic equation by factoring the quadratic expression)

$$(x+4)(x-2) = 0 \Rightarrow x = -4 \text{ or } x = 2$$

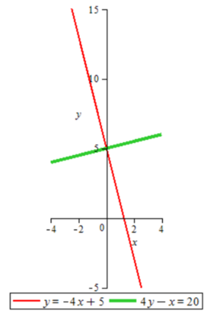
(Check that the answers found satisfy the original equation)

$$x = -4\text{: } \frac{(-4)^2 + 2(-4)}{(-4)-2} = \frac{8}{(-4)-2}? \Leftrightarrow \frac{8}{-6} = \frac{8}{-6}? \quad \textit{Yes!}$$

$$x = 2\text{: } \frac{2^2 + 2(2)}{2-2} = \frac{8}{2-2}? \Leftrightarrow \frac{8}{0} = \frac{8}{0}? \quad \textit{No! Denominator cannot be zero.}$$

Hence $\boldsymbol{x = -4}$.

10. (Must Know) Find an equation of the line perpendicular to the graph of the line $4y - x = 20$ and passing through $(2, -3)$. Sketch the graphs. [Answer: $y = -4x + 5$]



Solution:

(First, find the slope, m_1, of the given line by rewriting its equation into the slope-intercept form) $4y - x = 20 \Longrightarrow 4y = x + 20 \Longrightarrow y = \frac{1}{4}x + 5 \Longrightarrow m_1 = \frac{1}{4}$

(Find the slope m_2 of the line perpendicular to the given line and passing through (2, –3)) $m_1 m_2 = -1 \Longrightarrow m_2 = -1/m_1 = -4$

(Find the equation of the line by using the point-slope form and rewrite it in the slope intercept form)

$y - (-3) = -4(x - 2) \Longrightarrow y + 3 = -4x + 8 \Longrightarrow \boldsymbol{y = -4x + 5}$

(Sketch the graphs of the lines by using the point-slope form of their equations:

$4y - x = 20$ is a line with slope ¼ and y-intercept 5;

$y = -4x + 5$ is a line with slope –4 and y-intercept 5)

11. (Must Know) Given that $\log_a 2 = 1.5$ and $\log_a 3 = 0.4$, find $\log_a \left(\frac{12}{a}\right)$. [Answer: 2.4]

Solution:

$$\log_a \left(\frac{12}{a}\right) = \log_a 12 - \log_a a = \log_a(2^2 \times 3) - 1 = \log_a(2^2) + \log_a(3) - 1$$
$$= 2\log_a(2) + \log_a(3) - 1 = 2 \times 1.5 + 0.4 - 1 = \mathbf{2.4}$$

12. (Must Know) Solve the equation

(a) $e^x + e^{-x} = 6$. [Answer: $x = \ln(3 - 2\sqrt{2}), \ln(3 + 2\sqrt{2})$]

Solution:

(Rewrite the equation by using the substitution $y = e^x$ and simplify the result)

$$y = e^x \Longrightarrow y + y^{-1} = 6 \Longrightarrow y - 6 + \frac{1}{y} = 0 \Longrightarrow y^2 - 6y + 1 = 0$$

(For the resulting quadratic equation, the determinant $D = (-6)^2 - 4(1)(1) = 32$ is positive but not a perfect square, hence the roots are real and irrational : we cannot factor the quadratic expression and we use the quadratic formula)

$$y = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(1)}}{2(1)} = \frac{6 \pm \sqrt{32}}{2} = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$$

(We complete the solution by solving for x using the values we found for y)

$$\begin{cases} e^x = 3 - 2\sqrt{2} \Longrightarrow x = \ln\left(3 - 2\sqrt{2}\right) \\ e^x = 3 + 2\sqrt{2} \Longrightarrow x = \ln\left(3 + 2\sqrt{2}\right) \end{cases} \Longrightarrow \boldsymbol{x = \ln\left(3 - 2\sqrt{2}\right), \ln\left(3 + 2\sqrt{2}\right)}$$

(b) $\log_3(x-1) + \log_3(x+2) = 1$ [Answer: $x = \frac{-1+\sqrt{21}}{2} \approx 1.791$]

Solution:

(Rewrite and simplify the equation by using the properties of logarithmic function)

$$\log_3(x-1) + \log_3(x+2) = 1 \Longrightarrow \log_3[(x-1)(x+2)] = 1$$

$$\Longrightarrow (x-1)(x+2) = 3^1 \Longrightarrow x^2 + 2x - x - 2 = 3 \Longrightarrow x^2 + x - 5 = 0$$

(For the resulting quadratic equation, the determinant $D = 1^2 - 4(1)(-5) = 21$ is positive but not a perfect square, hence the roots are real and irrational : we cannot factor the quadratic expression and we use the quadratic formula)

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-5)}}{2(1)} = \frac{-1 \pm \sqrt{21}}{2}$$

(Check that the answers found satisfy the original equation)

$$x = \frac{-1+\sqrt{21}}{2}: \log_3\left(\frac{-1+\sqrt{21}}{2} - 1\right) + \log_3\left(\frac{-1+\sqrt{21}}{2} + 2\right) = 1?$$

$$\log_3\left(\frac{-3+\sqrt{21}}{2}\right) + \log_3\left(\frac{3+\sqrt{21}}{2}\right) = 1?$$

$$\log_3(0.7913) + \log_3(3.7913) \approx 1? \ \frac{\log 0.7913}{\log 3} + \frac{\log 3.7913}{\log 3} \approx 1? \quad Yes!$$

$$x = \frac{-1-\sqrt{21}}{2}: \log_3\left(\frac{-1-\sqrt{21}}{2} - 1\right) + \log_3\left(\frac{-1-\sqrt{21}}{2} + 2\right) = 1?$$

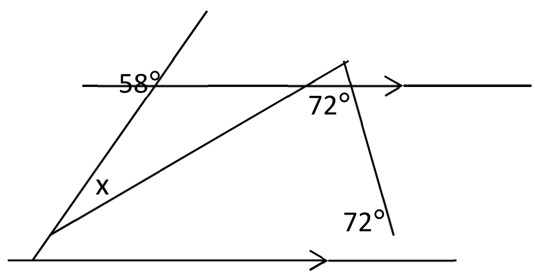
$$\log_3\left(\frac{-3-\sqrt{21}}{2}\right) + \log_3\left(\frac{3-\sqrt{21}}{2}\right) = 1?$$

$\log_3(-3.7913) + \log_3(-0.7913) \approx 1?$

No! The logarithmic function is not defined for negative numbers.

Hence $\boldsymbol{x} = \frac{\mathbf{-1+\sqrt{21}}}{\mathbf{2}} \approx \mathbf{1.791}$.

13. (Must Know) Find the unknown angle marked



[Answer: $x = 22°$]

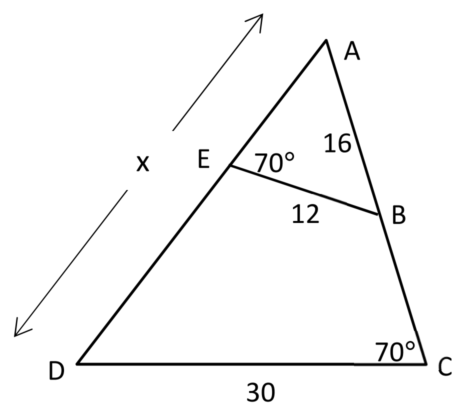
Solution:

$x + (180° - 72° - 72°) = 58°$

$\Longrightarrow x + 36° = 58°$

$\Longrightarrow x = 58° - 36° = \mathbf{22°}$

14. (Must Know) Find x.



[Answer: $x = 40$]

Solution:

Since $\angle ABE = \angle ADC$

$(= 180° - 70° - \angle A)$,

ΔABE is similar to ΔADC (as the corresponding interior angles match),

hence $\frac{AB}{BE} = \frac{AD}{DC} \Longrightarrow \frac{16}{12} = \frac{x}{30}$

$$\Longrightarrow x = \frac{30 \times 16}{12} = \mathbf{40}$$

15. (Must Know) If $\cos\theta = \frac{6}{7}$ and θ is an acute angle, find the other five trigonometric function values of θ.

[Answer: $\sin\theta = \frac{\sqrt{13}}{7}, \tan\theta = \frac{\sqrt{13}}{6}, \csc\theta = \frac{7\sqrt{13}}{13}, \sec\theta = \frac{7}{6}, \cot\theta = \frac{6\sqrt{13}}{13}$]

Solution:

Draw a right triangle and label one of the two interior acute angles as θ.

Since $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, label the side adjacent to θ as $b = 6$ and the hypotenuse opposite to the right angle as $c = 7$.

Apply the Pythagorean theorem to compute the side a opposite to θ :

$$a^2 + b^2 = c^2 \Rightarrow a^2 + 6^2 = 7^2 \Rightarrow a^2 = 49 - 36 = 13 \Rightarrow a = \sqrt{13}$$

Now we can find the remaining five trigonometric function values of θ by using the definitions:

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\mathbf{7}}{\mathbf{6}};$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{\mathbf{13}}}{\mathbf{7}}; \quad \csc\theta = \frac{1}{\sin\theta} = \frac{7}{\sqrt{13}} = \frac{\mathbf{7}\sqrt{\mathbf{13}}}{\mathbf{13}};$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{\mathbf{13}}}{\mathbf{6}}; \quad \cot\theta = \frac{1}{\tan\theta} = \frac{6}{\sqrt{13}} = \frac{\mathbf{6}\sqrt{\mathbf{13}}}{\mathbf{13}}.$$

16. (Nice To Know) Given $\cot\theta = -0.1611$ and $270° < \theta < 360°$, find θ.

[Answer: $\theta \approx 279.15°$]

Solution:

$$\tan\theta = \frac{1}{\cot\theta} = \frac{1}{-0.1611} \Longrightarrow \tan^{-1}\left(\frac{1}{-0.1611}\right) \approx -80.85°$$

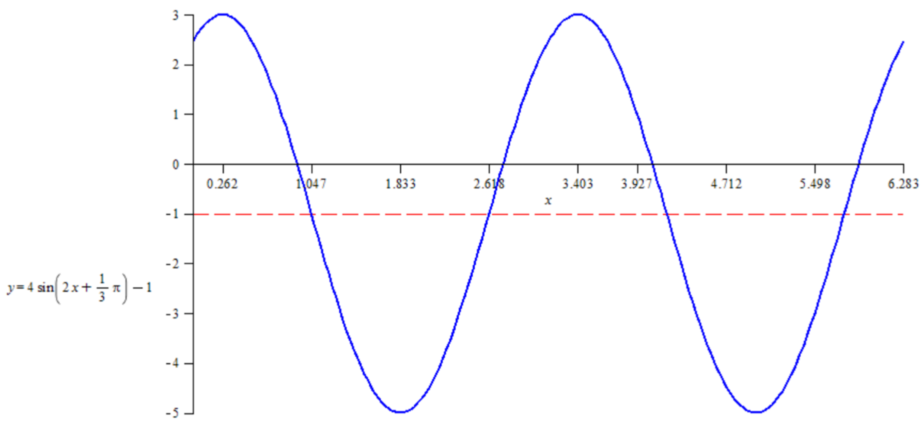
This angle terminates in quadrant IV with reference angle

$$\phi = 80.85°$$

Since $270° < \theta < 360°$, it is a positive angle that terminates in quadrant IV with the same reference angle:

$$\theta = 360° - 80.85° = \mathbf{279.15}°$$

17. (Must Know) Sketch $y = 4\sin\left(2x + \frac{\pi}{3}\right) - 1$ by finding the amplitude, the period, and the phase shift. [Answer: Amplitude = 4, period = π, phase shift = $-\frac{\pi}{6}$]



Solution:

We start with the characteristics of $y = \sin x$:

amplitude = 1, period = 2π, phase shift = 0.

The we make one change of the function each time and note the change of one the characteristics:

$y = \sin\left(x + \frac{\pi}{6}\right)$:

Graph shift horizontally π/6 to the left (phase shift = $-\frac{\pi}{6}$)

$y = \sin\left(2x + \frac{\pi}{3}\right) = \sin\left[2\left(x + \frac{\pi}{6}\right)\right]$:

Graph compressed horizontally by a factor of 2 (period = π)

$y = 4\sin\left(2x + \frac{\pi}{3}\right)$:

Graph stretched vertically by a factor or 4 (amplitude = 4)

$y = 4\sin\left(2x + \frac{\pi}{3}\right) - 1$:

Graph shifted vertically down by 1 unit.

Hence **amplitude = 4**, **period = $\boldsymbol{\pi}$**, **phase shift = $-\frac{\pi}{6}$**.

18. (Nice To Know) Prove the identity $(\sec\theta + \tan\theta)(1 - \sin\theta) = \cos\theta$.

Solution:

Left Hand Side (L.H.S.) = $(\sec\theta + \tan\theta)(1 - \sin\theta)$

$= \sec\theta - \sin\theta\sec\theta + \tan\theta - \sin\theta\tan\theta$

$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} + \tan\theta - \sin\theta \cdot \frac{\sin\theta}{\cos\theta}$

$$= \frac{1}{\cos\theta} - \cancel{\tan\theta} + \cancel{\tan\theta} - \frac{\sin^2\theta}{\cos\theta}$$

$$= \frac{1-\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta}$$

$$= \cos\theta = \text{Right Hand Side (R.H.S.)}$$

19. (Nice To Know) Simplify.

(a) $\frac{2\sin^2\theta+\sin\theta-3}{1-\sin\theta-\cos^2\theta}$. [Answer: $2+3\csc\theta$]

Solution:

$$\frac{2\sin^2\theta+\sin\theta-3}{1-\sin\theta-\cos^2\theta} = \frac{2\sin^2\theta+\sin\theta-3}{(1-\cos^2\theta)-\sin\theta} = \frac{2\sin^2\theta+\sin\theta-3}{\sin^2\theta-\sin\theta}$$

$$= \frac{\cancel{(\sin\theta-1)}(2\sin\theta+3)}{\sin\theta\,\cancel{(\sin\theta-1)}} = \frac{2\sin\theta}{\sin\theta} + \frac{3}{\sin\theta} = \mathbf{2+3\csc\boldsymbol{\theta}}$$

(b) $2\sin^2\frac{x}{2} + \cos x$. [Answer: 1]

Solution:

$$2\sin^2\frac{x}{2} + \cos x = \left[1-\cos\left(2\cdot\frac{x}{2}\right)\right] + \cos x = 1 - \cos x + \cos x = \mathbf{1}$$

20. (Nice To Know) Find the exact value of $\tan 15°$. [Answer: $2-\sqrt{3}$]

Solution:

$$\tan 15° = \tan(45°-30°) = \frac{\tan 45° - \tan 30°}{1+\tan 45°\tan 30°} = \frac{1-(1/\sqrt{3})}{1+(1)(1/\sqrt{3})}$$

$$= \frac{1-1/\sqrt{3}}{1+1/\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{3-2\sqrt{3}+1}{3-1} = \frac{4-2\sqrt{3}}{2}$$

$$= \mathbf{2-\sqrt{3}}$$

21. (Nice To Know) Solve the equation.

(a) $\cos\theta + \cos(2\theta) = 0$ for θ satisfying $0 \le \theta < 2\pi$. [Answer: $\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$]

Solution:

(Rewrite the equation in terms of only cos θ)

$$\cos\theta + (2\cos^2\theta - 1) = 0 \Longrightarrow 2\cos^2\theta + \cos\theta - 1 = 0$$

(This results in a quadratic equation in cos θ; since the discriminant

$D = 1^2 - 4(2)(-1) = 9 = 3^2$ is positive and a perfect square, we can solve for cos θ by factoring the quadratic expression and then solve for θ):

$(\cos\theta + 1)(2\cos\theta - 1) = 0 \Longrightarrow \cos\theta = -1 \text{ or } 1/2$

$$\begin{cases}\cos\theta = -1 \Longrightarrow \theta = \pi \\ \cos\theta = 1/2 \Longrightarrow \theta = \cos^{-1}(1/2) = \pi/3 \quad \text{or} \quad \theta = 2\pi - \pi/3 = 5\pi/3\end{cases}$$

Hence $\boldsymbol{\theta = \pi/3, \pi, 5\pi/3}$.

(b) $\tan^2 x + \sec x = 1$ in $[0, 2\pi)$. [Answer: $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$]

Solution:

(Rewrite the equation in terms of only sec *x*)

$(\sec^2 x - 1) + \sec x - 1 = 0 \Rightarrow \sec^2 x + \sec x - 2 = 0$

(This results in a quadratic equation in sec *x*; since the discriminant

$D = 1^2 - 4(1)(-2) = 9 = 3^2$ is positive and a perfect square, we can solve for sec *x* by factoring the quadratic expression and then solve for *x*):

$(\sec x - 1)(\sec x + 2) = 0 \Longrightarrow \sec x = 1 \text{ or } -2 \Rightarrow \cos x = 1 \text{ or } -1/2$

$$\begin{cases}\cos x = 1 \Longrightarrow x = 0 \\ \cos x = -1/2 \Longrightarrow x = \cos^{-1}(-1/2) = 2\pi/3 \quad \text{or} \quad x = \pi + (\pi - 2\pi/3) = 4\pi/3\end{cases}$$

Hence $\boldsymbol{x = 0, 2\pi/3, 4\pi/3}$.

(c) $\sin(3t + 2.55) = -1$ for (all solutions of) *t*.

[Answer: $t = \frac{1}{3}\left(\frac{3\pi}{2} - 2.55 + 2n\pi\right)$ for $n = 0, \pm 1, \pm 2, \cdots$]

Solution:

$$\sin(3t + 2.55) = -1 \Longrightarrow 3t + 2.55 = \frac{3\pi}{2} + 2n\pi \text{ (for } n = 0, \pm 1, \pm 2, \cdots)$$

$$\Longrightarrow 3t = \frac{3\pi}{2} - 2.55 + 2n\pi$$

$$\boldsymbol{\Longrightarrow t = \frac{1}{3}\left(\frac{3\pi}{2} - 2.55 + 2n\pi\right) \quad (\text{for } n = 0, \pm 1, \pm 2, \cdots)}$$

22. (Nice To Know) For the vectors $\boldsymbol{u} = [4\cos\theta \quad 4\sin\theta \quad 3]$ and $\boldsymbol{v} = [-4r\sin\theta \quad 4r\cos\theta \quad 0]$, evaluate $\|\boldsymbol{u} \times \boldsymbol{v}\|$. [Answer: $20|r|$]

Solution:

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4\cos\theta & 4\sin\theta & 3 \\ -4r\sin\theta & 4r\cos\theta & 0 \end{vmatrix}$$

$$= (4\sin\theta)(0)\boldsymbol{i} + (3)(-4r\sin\theta)\boldsymbol{j} + (4\cos\theta)(4r\cos\theta)\boldsymbol{k} - (4\sin\theta)(-4r\sin\theta)\boldsymbol{k} - (3)(4r\cos\theta)\boldsymbol{i} - (4\cos\theta)(0)\boldsymbol{j}$$

$$= (-12r\cos\theta)\boldsymbol{i} + (-12r\sin\theta)\boldsymbol{j} + (16r\cos^2\theta + 16r\sin^2\theta)\boldsymbol{k}$$

$$= (-12r\cos\theta)\boldsymbol{i} + (-12r\sin\theta)\boldsymbol{j} + (16r)\boldsymbol{k}$$

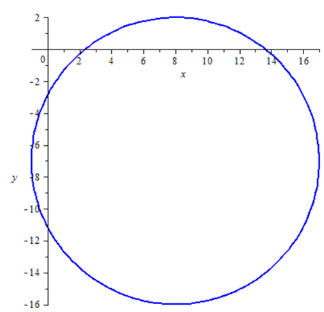
$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \sqrt{(-12r\cos\theta)^2 + (-12r\sin\theta)^2 + (16r)^2}$$

$$= \sqrt{144r^2\cos^2\theta + 144r^2\sin^2\theta + 256r^2}$$

$$= \sqrt{144r^2(\cos^2\theta + \sin^2\theta) + 256r^2} = \sqrt{144r^2 + 256r^2}$$

$$= \sqrt{400r^2} = \boldsymbol{20|r|}$$

23. (Must Know) For the circle $x^2 + y^2 - 16x + 14y + 32 = 0$, find the center and the radius, and then sketch the circle. [Answer: center: $(8, -7)$, radius = 9]



Solution:

(Convert the equation to standard form, identify the center and the radius of the circle and then we can sketch the graph)

$$x^2 + y^2 - 16x + 14y + 32 = 0$$

$$\Rightarrow \left[x^2 - 16x + \left(\frac{-16}{2}\right)^2\right] + \left[y^2 + 14y + \left(\frac{14}{2}\right)^2\right] = -32 + 64 + 49$$

$$\Rightarrow [x^2 - 16x + 64] + [y^2 + 14y + 49] = 81$$

$$\Rightarrow (x-8)^2 + (y+7)^2 = 9^2$$

Hence **the center is at** $(\mathbf{8}, -\mathbf{7})$ and **the radius is 9**.

24. (Nice To Know) For the function $y = -x^2 + 14x - 47$, find the vertex, the axis of symmetry, the intervals on which the function is increasing and decreasing, and the maximum or minimum value of the function, then graph the function.

[Answer: Vertex: $(7,2)$, axis of symmetry: $x = 7$, the function increases on the interval $(-\infty, 7)$ and decreases on the interval $(7, \infty)$, maximum value: 2; sketch the graph – a parabola that opens down with vertex $(7,2)$, x-intercepts $7 \pm \sqrt{2} \approx 5.586, 8.414$, and y-intercept -47]

Solution:

(Convert the equation to standard form to identify the vertex, the axis of symmetry, the monotonicity of the function and the maximum/minimum value)

$y = -x^2 + 14x - 47 \Rightarrow x^2 - 14x = -y - 47$

$\Rightarrow x^2 - 14x + \left(\frac{-14}{2}\right)^2 = -y - 47 + 49 \Rightarrow (x-7)^2 = -y + 2$

$\Rightarrow (x-7)^2 = -(y-2)$

Hence **the vertex is $(7,2)$** and **the axis of symmetry is $x = 7$.**

Since the coefficient of the first degree binomial (y – 2) is –1 < 0, the graph of the equation is a parabola that opens down, hence **the function increases on the interval $(-\infty, 7)$ and decreases on the interval $(7, \infty)$** and **the maximum value of the function is 2** (the y-coordinate of the vertex).

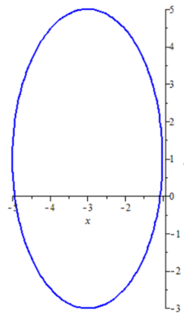
In addition to the information already found, we can sketch a more accurate graph by find the x- and the y-intercepts of the graph:

x-intercepts: (Set $y = 0$) $-x^2 + 14x - 47 = 0$

$\Longrightarrow x = \frac{-14 \pm \sqrt{14^2 - 4(-1)(-47)}}{2(-1)} = \frac{-14 \pm \sqrt{8}}{-2} = \frac{-14 \pm 2\sqrt{2}}{-2} = 7 \mp \sqrt{2} \approx 5.586, 8.414$

y-intercept: (Set $x = 0$) $y = -0^2 + 14(0) - 47 = -47$

25. (Nice To Know) For the ellipse $4x^2 + y^2 + 24x - 2y + 21 = 0$, find the center and the vertices. Then sketch the graph. [Answer: Center: $(-3,1)$, vertices: $(-3,-3), (-3,5)$]



Solution:

(Convert the equation to standard form to identify the vertex and the location of the vertices)

$$4x^2 + y^2 + 24x - 2y + 21 = 0 \Longrightarrow 4x^2 + 24x + y^2 - 2y = -21$$

$$\Longrightarrow 4\left[x^2 + 6x + \left(\frac{6}{2}\right)^2\right] + \left[y^2 - 2y + \left(\frac{-2}{2}\right)^2\right] = -21 + 4(9) + 1$$

$$\Longrightarrow 4[x^2 + 6x + 9] + [y^2 - 2y + 1] = 16 \Longrightarrow 4(x+3)^2 + (y-1)^2 = 16$$

$$\Longrightarrow \frac{4(x+3)^2}{16} + \frac{(y-1)^2}{16} = 1 \Longrightarrow \frac{(x+3)^2}{4} + \frac{(y-1)^2}{16} = 1$$

$$\Longrightarrow a = \sqrt{16} = 4; b = \sqrt{4} = 2$$

Hence **the center is located at** $(\mathbf{-3, 1})$ and the two vertices are vertically 4 units from the center, that is, **the vertices are located at** $(\mathbf{-3, -3})$ **and** $(\mathbf{-3, 5})$.

26. (Nice To Know) Expand $\left(2t + \frac{3}{t}\right)^4$. [Answer: $16t^4 + 96t^2 + 216 + \frac{216}{t^2} + \frac{81}{t^4}$]

Solution:

(Apply the Binomial Theorem)

$$\left(2t + \frac{3}{t}\right)^4$$

$$= C(4,0)(2t)^4 + C(4,1)(2t)^3\left(\frac{3}{t}\right) + C(4,2)(2t)^2\left(\frac{3}{t}\right)^2 + C(4,3)(2t)\left(\frac{3}{t}\right)^3 + C(4,4)\left(\frac{3}{t}\right)^4$$

$$= (1)(16t^4) + (4)(8t^3)\left(\frac{3}{t}\right) + (6)(4t^2)\left(\frac{9}{t^2}\right) + (4)(2t)\left(\frac{27}{t^3}\right) + (1)\left(\frac{81}{t^4}\right)$$

$$= \mathbf{16t^4 + 96t^2 + 216 + \frac{216}{t^2} + \frac{81}{t^4}}$$

27. (Must Know) Evaluate the limit:

(a) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$ [Answer: 5]

Solution:

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x\to 2} (x+3) = 2+3 = \mathbf{5}$$

(b) $\lim_{x\to\infty} \frac{\sqrt{4x^2+5}}{x+1}$ [Answer: 2]

Solution:

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+5}}{x+1} = \lim_{x\to\infty} \frac{\sqrt{x^2[4+(5/x^2)]}}{x+1} = \lim_{x\to\infty} \frac{\sqrt{x^2}\times\sqrt{4+(5/x^2)}}{x+1}$$

$$= \lim_{x\to\infty} \frac{x\times\sqrt{4+(5/x^2)}}{x+1} = \lim_{x\to\infty} \frac{\sqrt{4+(5/x^2)}}{(x+1)/x} = \lim_{x\to\infty} \frac{\sqrt{4+(5/x^2)}}{1+(1/x)}$$

$$= \frac{\sqrt{4}}{1} = \mathbf{2}$$

28. Evaluate and simplify the following derivatives.

(a) (Nice To Know) $\frac{d}{dx}[\sin(\pi x) + \sin^{-1}(x^2)]$ [Answer: $\pi\cos(\pi x) + \frac{2x}{\sqrt{1-x^4}}$]

Solution:

$$\frac{d}{dx}[\sin(\pi x) + \sin^{-1}(x^2)] = \frac{d}{dx}\sin(\pi x) + \frac{d}{dx}\sin^{-1}(x^2)$$

$$= \cos(\pi x)\frac{d}{dx}(\pi x) + \frac{1}{\sqrt{1-(x^2)^2}}\frac{d}{dx}(x^2)$$

$$= \boldsymbol{\pi\cos(\pi x) + \frac{2x}{\sqrt{1-x^4}}}$$

(b) (Must Know) $\frac{d}{dx}[e^{2x}\cos 5 + 16\sin(3x-10)]$ [Answer: $2e^{2x}\cos 5 + 48\cos(3x-10)$]

Solution:

$$\frac{d}{dx}[e^{2x}\cos 5 + 16\sin(3x-10)] = (\cos 5)\frac{d}{dx}[e^{2x}] + 16\frac{d}{dx}[\sin(3x-10)]$$

$$= (\cos 5)e^{2x}\frac{d}{dx}[2x] + 16\cos(3x-10)\frac{d}{dx}[3x-10]$$

$$= \mathbf{2(\cos 5)e^{2x} + 48\cos(3x-10)}$$

(c) (Must Know) $\frac{d}{dx}\ln(\cos 2x)$ [Answer: $-2\tan 2x$]

Solution:

$$\frac{d}{dx}\ln(\cos 2x) = \frac{1}{\cos(2x)}\frac{d}{dx}[\cos(2x)] = \frac{1}{\cos(2x)}[-\sin(2x)]\frac{d}{dx}[2x]$$

$$= -\frac{2\sin(2x)}{\cos(2x)} = \mathbf{-2\tan(2x)}$$

(d) (Must Know) $\frac{d}{dx}[\sec^2(3x) + \tan(4x)]$

[Answer: $6\sec^2(3x)\tan(3x) + 4\sec^2(4x)$]

Solution:

$$\frac{d}{dx}[\sec^2(3x) + \tan(4x)] = 2\sec(3x)\frac{d}{dx}[\sec(3x)] + \sec^2(4x)\frac{d}{dx}[4x]$$

$$= 2\sec(3x) \times \sec(3x)\tan(3x)\frac{d}{dx}[3x] + 4\sec^2(4x)$$

$$= \mathbf{6\sec^2(3x)\tan(3x) + 4\sec^2(4x)}$$

(e) (Nice To Know) $\frac{d}{ds}[2s\tan^{-1}s - \ln(1+s^2)]$ [Answer:$2\tan^{-1}s$]

Solution:

$$\frac{d}{ds}[2s\tan^{-1}s - \ln(1+s^2)] = 2\frac{d}{ds}[s\tan^{-1}s] - \frac{d}{ds}\ln(1+s^2)$$

$$= 2\left[1 \times \tan^{-1}s + s \times \frac{1}{1+s^2}\right] - \frac{1}{1+s^2}\frac{d}{ds}(1+s^2)$$

$$= 2\left[\tan^{-1}s + \frac{s}{1+s^2}\right] - \frac{1}{1+s^2} \times 2s$$

$$= 2\tan^{-1}s + \frac{2s}{1+s^2} - \frac{2s}{1+s^2} = \mathbf{2\tan^{-1}s}$$

(f) (Must Know) $\frac{d}{dx}\left(\frac{3-2x}{x^2+2}\right)$ [Answer: $\frac{2(x^2-3x-2)}{(x^2+2)^2}$]

Solution:

$$\frac{d}{dx}\left(\frac{3-2x}{x^2+2}\right) = \frac{(x^2+2)\frac{d}{dx}(3-2x) - (3-2x)\frac{d}{dx}(x^2+2)}{(x^2+2)^2}$$

$$= \frac{-2(x^2+2) - 2x(3-2x)}{(x^2+2)^2} = \frac{-2x^2 - 4 - 6x + 4x^2}{(x^2+2)^2}$$

$$= \frac{2x^2 - 6x - 4}{(x^2+2)^2} = \frac{\mathbf{2(x^2 - 3x - 2)}}{\mathbf{(x^2+2)^2}}$$

29. (Must Know) Find an equation for the tangent line at point (2, 4) on the curve $x^3 + y^3 = 9xy$.

[Answer: $y = \frac{4}{5}x + \frac{12}{5}$ or $4x - 5y = -12$]

Solution:

(First use implicit differentiation to find the slope of the curve m at the point (2, 4))

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(9xy) \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 9\left(1 \times y + x \times \frac{dy}{dx}\right)$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 9y + 9x\frac{dy}{dx}$$

$$\Rightarrow 3 \times 2^2 + 3 \times 4^2 \times m = 9 \times 4 + 9 \times 2 \times m$$

$$\Rightarrow 12 + 48m = 36 + 18m \Rightarrow 30m = 24 \Rightarrow m = \frac{24}{30} = \frac{4}{5}$$

(Use the point-slope form to find the equation of the tangent line)

$$y - 4 = \frac{4}{5}(x-2) \Longrightarrow y - 4 = \frac{4}{5}x - \frac{8}{5}$$

$$\Longrightarrow \boldsymbol{y = \frac{4}{5}x + \frac{12}{5}} \quad \text{or} \quad \mathbf{4x - 5y = -12}$$

30. (Nice To Know) For what positive values of x is the function $f(x) = \frac{\sqrt{x}}{x+1}$ decreasing?

[Answer: $x > 1$]

Solution:

(Check the domain of the function: denominator cannot be zero; radicand must be non-negative)

$$x + 1 \neq 0 \Leftrightarrow x \neq -1; \quad x > 0$$

($f(x)$ is decreasing when $f'(x) < 0$)

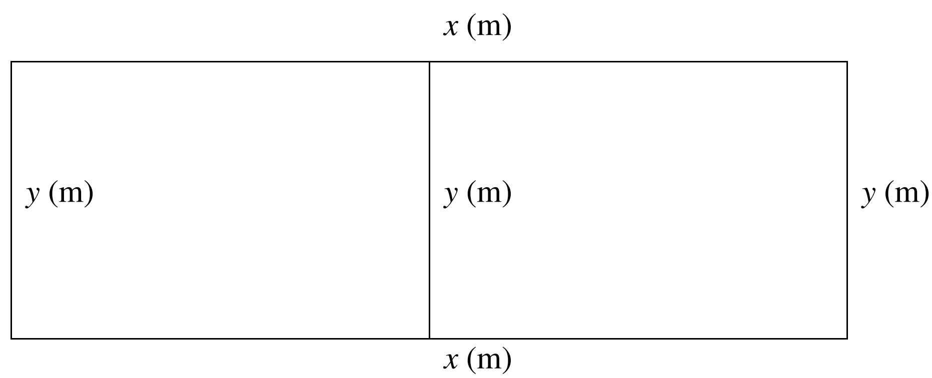
$$f'(x) = \frac{(x+1)\frac{1}{2\sqrt{x}} - \sqrt{x}\cdot 1}{(x+1)^2} = \frac{x+1-2x}{2(x+1)^2\sqrt{x}} = \frac{1-x}{2(x+1)^2\sqrt{x}}$$

Interval	(0, 1)	(1, ∞)
Test Point (one possibility)	0.5	2
Inequality $\frac{1-x}{2(x+1)^2\sqrt{x}} < 0$ Satisfied?	$\frac{(+)}{(+)(+)(+)} < 0?$	$\frac{(-)}{(+)(+)(+)} < 0?$
Part of the Solution?	No	Yes

Hence $f(x) = \frac{\sqrt{x}}{x+1}$ is decreasing for $\boldsymbol{x > 1}$.

31. (Must Know) A rectangular field is to be fenced and then divided in half by a fence parallel to two opposite sides. If a total of 6000 m of fencing is used, what is the maximum area that can be fenced? [Answer: 1.5×10^6 m^2]

Solution:



Let the dimensions of the rectangular field be x(m) × y(m) and the divider be of length y(m). Since a total of 6000 m of fencing is used,

$$2x + 3y = 6000 \Longrightarrow 3y = 6000 - 2x \Longrightarrow y = 2000 - \frac{2}{3}x$$

(Rewrite the area of the rectangular field A as a function of one variable x)

$$A = xy = x\left(2000 - \frac{2}{3}x\right) = 2000x - \frac{2}{3}x^2$$

(Find the critical number(s) by setting the first derivative to zero)

$$\frac{dA}{dx} = 0 \Longrightarrow 2000 - \frac{4}{3}x = 0 \Longrightarrow \frac{4}{3}x = 2000 \Longrightarrow x = 2000 \times \frac{3}{4} = 1500$$

(Since there is only one critical number, this must lead to the maximum area)

$$\text{Maximum Area} = 2000(1500) - \frac{2}{3}(1500)^2 = 1500000 = \mathbf{1.5 \times 10^6\ m^2}$$

32. (Nice To Know) For the first 12 s after launch, the height *s* (in m) of a certain rocket is given by $s = 10\sqrt{t^4 + 25} - 50$. Find the vertical acceleration of the rocket when $t = 10.0$ s.

[Answer: $\approx 20.1\ \text{m/s}^2$]

Solution:

$$\text{Velocity } v = \frac{ds}{dt} = 10 \times \frac{1}{2}(t^4 + 25)^{-1/2} \times 4t^3 = \frac{20t^3}{\sqrt{t^4+25}}$$

$$\text{Acceleration } a = \frac{dv}{dt}$$

$$= 20 \times \frac{1}{t^4 + 25}\left[\sqrt{t^4 + 25} \times 3t^2 - t^3 \times \frac{1}{2}(t^4 + 25)^{-1/2} \times 4t^3\right]$$

$$= \frac{20}{t^4 + 25}\left[3t^2\sqrt{t^4 + 25} - 2t^6(t^4 + 25)^{-1/2}\right]$$

$$a(10) = \frac{20}{10^4 + 25}\left[3(10^2)\sqrt{10^4 + 25} - 2(10^6)(10^4 + 25)^{-1/2}\right]$$

$$\approx \mathbf{20.1\ m/s^2}$$

33. (Nice To Know) Find the magnitude and direction of the acceleration when $t = 2$ for an object that is moving such that its *x*- and *y*-coordinates of position are given by $x = t^3$ and $y = 1 - t^2$. [Answer: $a = 2\sqrt{37} \approx 12.2, \theta_a = \tan^{-1}\left(-\frac{1}{6}\right) \approx -9.5°$]

Solution:

$$x = t^3 \Rightarrow \frac{dx}{dt} = 3t^2 \Rightarrow \frac{d^2x}{dt^2} = 6t \Rightarrow a_x|_{t=2} = 12$$

$$y = 1 - t^2 \Rightarrow \frac{dy}{dt} = -2t \Rightarrow \frac{d^2y}{dt^2} = -2 \Rightarrow a_y|_{t=2} = -2$$

$$\text{Magnitude: } a = \sqrt{12^2 + (-2)^2} = \sqrt{148} = \sqrt{4 \times 37} = \mathbf{2\sqrt{37}} \approx \mathbf{12.2}$$

$$\text{Direction: } \left\{\begin{array}{c} \tan\theta_a = \frac{a_y}{a_x} = \frac{-2}{12} = -\frac{1}{6} \\ a_x > 0 \text{ and } a_y < 0 \Rightarrow -\frac{\pi}{2} < \theta_a < 0 \end{array}\right\}$$

$$\Rightarrow \theta_a = \mathbf{tan^{-1}}\left(-\frac{\mathbf{1}}{\mathbf{6}}\right) \approx \mathbf{-9.5^\circ}$$

34. (Nice To Know) Find the first two non-zero terms of the Maclaurin series of $\tan x$.

[Answer: $x + \frac{1}{3}x^3 + \cdots$]

Solution:

$$f(x) = \tan x \Rightarrow f(0) = \tan 0 = 0;$$

$$f'(x) = \sec^2 x \Rightarrow f'(0) = \sec^2 0 = 1;$$

$$f''(x) = 2\sec x \times \sec x \tan x = 2\sec^2 x \tan x \Rightarrow f''(0) = 2\sec^2 0 \tan 0 = 0;$$

$$f'''(x) = 2[2\sec x\,(\sec x \tan x) \times \tan x + \sec^2 x \times \sec^2 x]$$

$$= 2\sec^2 x\,[2\tan^2 x + \sec^2 x]$$

$$\Rightarrow f'''(0) = 2\sec^2 0\,[2\tan^2 0 + \sec^2 0] = 2(0+1) = 2$$

Hence $\tan x = 0 + 1 \cdot x + \frac{0}{2!} \cdot x^2 + \frac{2}{3!} \cdot x^3 + \cdots = \boldsymbol{x + \frac{1}{3}x^3} + \cdots$

35. (Nice To Know) Find $P_2(x)$, the Taylor polynomial of degree 2, for $\sin x$ centered at $c = \frac{\pi}{6}$.

[Answer: $P_2(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2$]

Solution:

$$f(x) = \sin x \Rightarrow f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2};$$

$$f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3};$$

$$f''(x) = -\sin x \Rightarrow f''^{\left(\frac{\pi}{6}\right)} = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2};$$

$$\Rightarrow P_2(x) = \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right) + \frac{1}{2!} \times \left(-\frac{1}{2}\right)\left(x - \frac{\pi}{6}\right)^2$$

$$= \boldsymbol{\frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2}$$

36. Evaluate (by using the method specified, if any) and simplify the following.

(a) (Must Know) $e^{\int \frac{1+3x}{x}dx}$ [Answer: $C|x|e^{3x}$]

Solution:

$$e^{\int \frac{1+3x}{x}dx} = e^{\int \left(\frac{1}{x}+3\right)dx} = e^{\ln|x|+3x+C_1} = e^{C_1}e^{\ln|x|}e^{3x} = \boldsymbol{C|x|e^{3x}}$$

(b) (Must Know) $\int_0^2 \frac{2x}{(x^2+1)^3}\, dx$ [Answer: $\frac{12}{25}$]

Solution:

(Integrate by substitution)

Let $u = x^2 + 1$, then $du = 2x\, dx$; $x = 0$, $u = 1$; $x = 2$, $u = 5$; hence

$$\int_0^2 \frac{2x}{(x^2+1)^3}\, dx = \int_1^5 \frac{du}{u^3} = \left[-\frac{1}{2u^2}\right]_1^5 = \left(-\frac{1}{50}\right) - \left(-\frac{1}{2}\right)$$

$$= \frac{-1+25}{50} = \frac{24}{50} = \boldsymbol{\frac{12}{25}}$$

(c) (Nice To Know) $\int \frac{3x+1}{x^2+9}\, dx$ [Answer: $\frac{3}{2}\ln(x^2+9) + \frac{1}{3}\tan^{-1}\frac{x}{3} + C$]

Solution:

$$\int \frac{3x+1}{x^2+9}\, dx = 3\int \frac{x}{x^2+9} dx + \int \frac{1}{x^2+9} dx$$

(For the first integral, we evaluate by using the substitution $u = x^2 + 9$)

$$u = x^2 + 9 \Longrightarrow du = 2x\, dx \Longrightarrow \int \frac{x}{x^2+9} dx = \int \frac{1}{u}\cdot\frac{1}{2} du = \frac{1}{2}\ln|u| + C_1$$

Hence

$$\int \frac{3x+1}{x^2+9}\, dx = 3\left[\frac{1}{2}\ln|u| + C_1\right] + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C_2$$

$$= \boldsymbol{\frac{3}{2}\ln(x^2+9) + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C}$$

(d) (Must Know) $\int \sin^2\varphi \cos^3\varphi\; d\varphi$ [Answer: $\frac{1}{3}\sin^3\varphi - \frac{1}{5}\sin^5\varphi + C$]

Solution:

(Integrate by substitution) $u = \sin\varphi \Rightarrow du = \cos\varphi\, d\varphi$

$$\Rightarrow \int \sin^2\varphi \cos^3\varphi\; d\varphi = \int \sin^2\varphi \cos^2\varphi \cos\varphi\; d\varphi$$

$$= \int \sin^2\varphi\,(1-\sin^2\varphi)\cos\varphi\; d\varphi = \int u^2(1-u^2)\, du$$

$$= \int (u^2 - u^4)\, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \boldsymbol{\frac{1}{3}\sin^3\varphi - \frac{1}{5}\sin^5\varphi + C}$$

(e) (Must Know) $\int_{-\pi/6}^{\pi/6} \cos^2 3\theta \; d\theta$ [Answer: $\frac{\pi}{6}$]

Solution:

$$\int_{-\pi/6}^{\pi/6} \cos^2 3\theta \; d\theta = \int_{-\pi/6}^{\pi/6} \frac{1}{2}[1+\cos(6\theta)]\, d\theta = \frac{1}{2}\left[\theta + \frac{1}{6}\sin(6\theta)\right]_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{2}\left\{\left[\frac{\pi}{6} + \frac{1}{6}\sin\left(\frac{6\pi}{6}\right)\right] - \left[\left(-\frac{\pi}{6}\right) + \frac{1}{6}\sin\left(-\frac{6\pi}{6}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\pi}{6} + 0\right] - \left[\left(-\frac{\pi}{6}\right) + 0\right]\right\} = \boldsymbol{\frac{\pi}{6}}$$

(f) (Must Know) (Integration by parts) $\int xe^{2x}\, dx$ [Answer: $\left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} + C$]

Solution:

$$\int xe^{2x}\, dx$$

$$= +(x)\left(\frac{1}{2}e^{2x}\right) - (1)\left(\frac{1}{4}e^{2x}\right) + C$$

$$= \boldsymbol{\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C}$$

or $\boldsymbol{\left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} + C}$

D	I
x	e^{2x}
1	$\frac{1}{2}e^{2x}$
0	$\frac{1}{4}e^{2x}$

(g) (Must Know) (Integration by parts) $\int x^2 \sin(2x)\, dx$

[Answer: $-\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$]

Solution:

$$\int x^2 \sin(2x)\, dx$$

$$= +(x^2)\left[-\frac{1}{2}\cos(2x)\right] - (2x)\left[-\frac{1}{4}\sin(2x)\right]$$

$$+(2)\left[\frac{1}{8}\cos(2x)\right] + C$$

$$= \boldsymbol{-\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C}$$

D	I
x^2	$\sin(2x)$
$2x$	$-\frac{1}{2}\cos(2x)$
2	$-\frac{1}{4}\sin(2x)$
0	$\frac{1}{8}\cos(2x)$

(h) (Nice To Know) (Trigonometric substitution) $\int \frac{1}{x^2\sqrt{9x^2-4}}\, dx$ [Answer: $\frac{\sqrt{9x^2-4}}{4x} + C$]

Solution:

$3x = 2\sec\theta \Rightarrow x = \frac{2}{3}\sec\theta \Rightarrow dx = \frac{2}{3}\sec\theta\tan\theta \; d\theta$, hence

$$\int \frac{1}{x^2\sqrt{9x^2-4}}\,dx = \int \frac{1}{\left(\frac{2}{3}\sec\theta\right)^2\sqrt{9\left(\frac{2}{3}\sec\theta\right)^2-4}}\cdot\frac{2}{3}\sec\theta\tan\theta\ d\theta$$

$$= \int \frac{9}{4\sec^2\theta\sqrt{4\sec^2\theta-4}}\cdot\frac{2}{3}\sec\theta\tan\theta\ d\theta$$

$$= \int \frac{3\tan\theta}{2\sec\theta\sqrt{4(\sec^2\theta-1)}}\,d\theta = \int \frac{3\tan\theta}{2\sec\theta\cdot 2\tan\theta}\,d\theta = \frac{3}{4}\int \cos\theta\ d\theta$$

$$= \frac{3}{4}\sin\theta + C = \frac{3}{4}\cdot\frac{\sqrt{9x^2-4}}{3x} + C = \boldsymbol{\frac{\sqrt{9x^2-4}}{4x} + C}$$

(i) (Nice To Know) (Partial fractions) $\int \frac{4x+4}{x^3+4x}\,dx$

[Answer: $\ln|x| - \frac{1}{2}\ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C$ or $\frac{1}{2}\ln\left(\frac{x^2}{x^2+4}\right) + 2\tan^{-1}\left(\frac{x}{2}\right) + C$]

Solution:

(Find the form of the partial fraction decomposition of the integrand)

$$\frac{4x+4}{x^3+4x} = \frac{4x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

(Find the numerical coefficients of the partial fraction decomposition)

$$x(x^2+4)\left[\frac{4x+4}{x(x^2+4)}\right] = \left[\frac{A}{x} + \frac{Bx+C}{x^2+4}\right] x(x^2+4)$$

$$\Rightarrow 4x+4 = A(x^2+4) + (Bx+C)x$$

$$x = 0: 4 = 4A \Rightarrow A = 1$$

$$\left.\begin{cases} x = 1: 8 = 5A + B + C \Rightarrow B + C = 3 \\ x = -1: 0 = 5A + B - C \Longrightarrow B - C = -5 \end{cases}\right\} \Rightarrow B = -1, C = 4$$

(Evaluate the integral by integrating term by term)

$$\int \left(\frac{1}{x} + \frac{-x+4}{x^2+4}\right) dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+4}\,dx + \int \frac{4}{x^2+4}\,dx$$

(We have standard integration formula for the first and the third integrals; for the second integral, we integrate by substitution)

$$\int \frac{1}{x}\,dx = \ln|x| + C_1;$$

$$u = x^2 + 4 \Rightarrow du = 2x\,dx$$

$$\Rightarrow \int \frac{x}{x^2+4}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}du = \frac{1}{2}\ln|u| + C_2 = \frac{1}{2}\ln(x^2+4) + C_2;$$

$$\int \frac{4}{x^2+4}\,dx = 4\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C_3 = 2\tan^{-1}\left(\frac{x}{2}\right) + C_3;$$ hence

$$\int \frac{4x+4}{x^3+4x}\,dx = \ln|x| + C_1 - \left[\frac{1}{2}\ln(x^2+4) + C_2\right] + 2\tan^{-1}\left(\frac{x}{2}\right) + C_3$$

$$= \mathbf{\ln|x| - \frac{1}{2}\ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C}$$

37. (Nice To Know) The velocity of a robot arm is $v = t\sqrt{9-t^2}$. Find the expression for the displacement as a function of time if $s = 0$ cm when $t = 0$ s.

[Answer: $s = 9 - \frac{1}{3}(9-t^2)^{3/2}$]

Solution:

$$s = \int v(t)\,dt = \int t\sqrt{9-t^2}dt = \int \sqrt{9-t^2}\,(t\,dt)$$

$$= \left(u = 9 - t^2 \Rightarrow du = -2t\,dt \Rightarrow t\,dt = -\frac{1}{2}du\right) = \int \sqrt{u}\times -\frac{1}{2}du$$

$$= -\frac{1}{2}\times\frac{2}{3}u^{3/2} + C = -\frac{1}{3}(9-t^2)^{3/2} + C$$

$t = 0$,, $s = 0$: $0 = -\frac{1}{3}(9-0^2)^{3/2} + C \Rightarrow C = \frac{1}{3}\times 27 = 9$,

hence $\mathbf{s = 9 - \frac{1}{3}(9-t^2)^{3/2}}$

38. (Must Know) Find the area between $y = x^2$ and $y = x + 2$. [Answer: $\frac{9}{2}$]

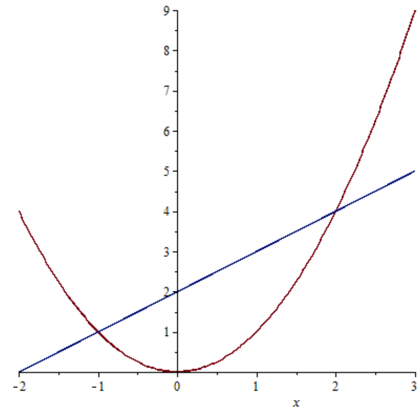
Solution:

(Find the x-coordinates of the intersection of the two curves)

$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1, 2$

(Find the area by evaluating a definite integral)



$$A = \int_{-1}^{2} [(x+2) - x^2]\, dx$$

$$= \int_{-1}^{2} (2 + x - x^2)\, dx = \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{2}$$

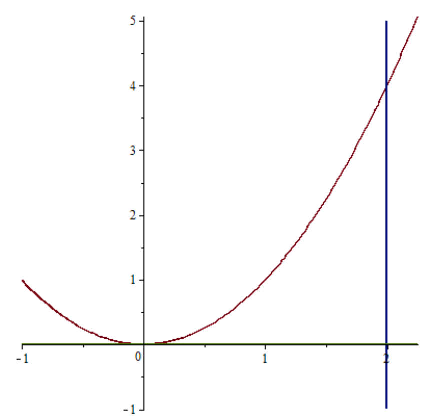
$$= \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \mathbf{\frac{9}{2}}$$

39. (Nice To Know) Find the volume of the solid generated by revolving the first-quadrant region bounded by $y = x^2, x = 2$, and $y = 0$ about

(a) the x-axis [Answer: $\frac{32\pi}{5}$]

Solution:

(Use the Disk Method)



$$V = \int_{0}^{2} \pi (x^2)^2\, dx = \pi \int_{0}^{2} x^4\, dx$$

$$= \frac{\pi}{5}[x^5]_0^2 = \frac{\pi}{5}(32 - 0) = \mathbf{\frac{32}{5}\pi}$$

(b) the y-axis [Answer: 8π]

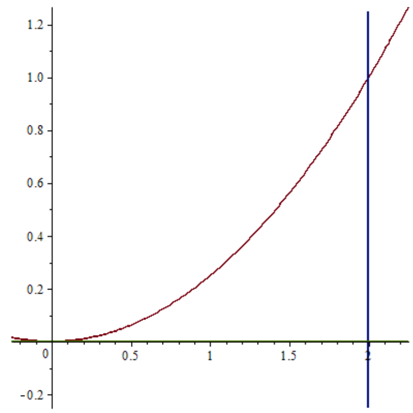
Solution:

(Use the Shell Method)

$$V = \int_{0}^{2} 2\pi x(x^2)\, dx = 2\pi \times \frac{1}{4}[x^4]_0^2 = \frac{\pi}{2}[16 - 0] = \mathbf{8\pi}$$

40. (Nice To Know) Find the coordinates of the centroid of a flat plate that covers the region bounded by $y = \frac{1}{4}x^2, y = 0$ and $x = 2$. [Answer: $(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{3}{10}\right)$]

Solution:



(Find the area)

$$A = \int_0^2 \frac{1}{4}x^2\,dx = \frac{1}{12}[x^3]_0^2$$

$$= \frac{1}{12}(8 - 0) = \frac{8}{12} = \frac{2}{3}$$

(Compute the first moments about the *x*- and the *y*-axes)

$$y = \frac{1}{4}x^2 \Rightarrow x^2 = 4y \Rightarrow x = \pm 2\sqrt{y}$$

$$M_x = \int_0^1 \left(2 - 2\sqrt{y}\right) \times y\,dy = 2\int_0^2 \left(y - y^{3/2}\right)dx = 2\left[\frac{1}{2}y^2 - \frac{2}{5}x^{5/2}\right]_0^1$$

$$= 2\left[\left(\frac{1}{2} - \frac{2}{5}\right) - (0 - 0)\right] = \frac{1}{5};$$

$$M_y = \int_0^2 x \times \frac{1}{4}x^2\,dx = \frac{1}{4}\int_0^2 x^3\,dx = \frac{1}{4} \times \frac{1}{4}[x^4]_0^2 = \frac{1}{16}(16 - 0) = 1;$$

Hence centroid $(\bar{x}, \bar{y}) = \left(\frac{M_y}{A}, \frac{M_x}{A}\right) = \left(\frac{1}{2/3}, \frac{1/5}{2/3}\right) = \left(\frac{\mathbf{3}}{\mathbf{2}}, \frac{\mathbf{3}}{\mathbf{10}}\right)$.

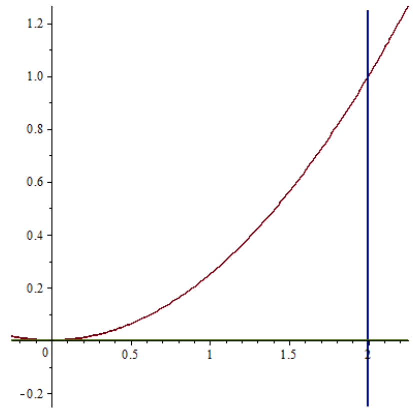
41. (Nice To Know) Find the moment of inertia of a flat plate (with uniform density *k*) that covers the region bounded by $y = \frac{1}{4}x^2, y = 0$ and $x = 2$ with respect to the *y*-axis.

[Answer: $\frac{8k}{5}$]

Solution:

$$I_y = \int_0^2 x^2 \times \left(\frac{k}{4}x^2\right) dx = \frac{k}{4}\int_0^2 x^4\, dx$$

$$= \frac{k}{4} \times \frac{1}{5}[x^5]_0^2 = \frac{k}{20}[32 - 0] = \boldsymbol{\frac{8k}{5}}$$



42. (Nice To Know) Solve the differential equation subject to the given condition:

$$xy\, dx + (x^2 + 1)\, dy = 0; \quad x = 0 \text{ when } y = e$$

[Answer: $y^2(x^2+1) = e^2$, or $y = \frac{e}{\sqrt{x^2+1}}$]

Solution:

(Separate the variables) $xy\, dx + (x^2+1)\, dy = 0 \Rightarrow \frac{x}{x^2+1}\, dx + \frac{1}{y}\, dy = 0$

(Integrate each term) $\int \frac{x}{x^2+1}\, dx + \int \frac{1}{y}\, dy = \int 0$

(For the first integral on the left hand side, use the substitution

$u = x^2 + 1 \Rightarrow du = 2x\, dx \Rightarrow x\, dx = \frac{1}{2} du$)

$$\int \frac{1}{u}\cdot\frac{1}{2}\, du + \int \frac{1}{y}\, dy = \int 0 \quad \Longrightarrow \frac{1}{2}\ln|u| + \ln|y| = C$$

$$\Longrightarrow \frac{1}{2}\ln|x^2+1| + \ln|y| = C$$

(Use the initial condition to determine the value of C, then simplify to get the answer)

$$x = 0, y = e\text{: } \frac{1}{2}\ln|0^2+1| + \ln|e| = C \Longrightarrow C = \frac{1}{2}\ln 1 + \ln e = 0 + 1 = 1$$

$$\Rightarrow \frac{1}{2}\ln(x^2+1) + \ln y = 1 \Rightarrow \ln y = 1 - \frac{1}{2}\ln(x^2+1) \Rightarrow y = e^{1-\frac{1}{2}\ln(x^2+1)}$$

$$\Rightarrow y = e^1 e^{-\frac{1}{2}\ln(x^2+1)} \Rightarrow y = e \cdot e^{\ln\left[(x^2+1)^{-1/2}\right]} \Rightarrow y = e \cdot (x^2+1)^{-1/2}$$

$$\Rightarrow \boldsymbol{y = \frac{e}{\sqrt{x^2+1}}}$$

43. (Nice To Know) Find the general solution of the differential equation $x\frac{dy}{dx} + 2y = 4$.

[Answer: $y = 2 + \frac{C}{x^2}$]

Solution:

(We solve this first-order linear DE by using the 7-step process)

(Step 1) $x\frac{dy}{dx} + 2y = 4 \Longrightarrow x\,dy + 2y\,dx = 4\,dx \Rightarrow dy + \frac{2}{x}\,y\,dx = \frac{4}{x}\,dx$

(Step 2) $\int P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x|$

(Step 3) Integrating Factor (I.F.) $= e^{2\ln|x|} = e^{\ln(|x|^2)} = |x|^2 = x^2$

(Step 4) $x^2\left(dy + \frac{2}{x}\,y\,dx\right) = x^2\left(\frac{4}{x}\,dx\right) \Rightarrow x^2\,dy + 2xy\,dx = 4x\,dx$

(Step 5) $d(x^2 \cdot y) = 4x\,dx$

(Step 6) $\int d(x^2 \cdot y) = \int 4x\,dx \Longrightarrow x^2 y = 2x^2 + C \Rightarrow \boldsymbol{y = 2 + \frac{C}{x^2}}$

(Step 7) With no additional information, we do not need to do this step.

44. (Nice To Know) Find the general solution of the differential equation

$$D^2y - 2Dy - 8y = 4e^{-2x}.$$

[Answer: $y = C_1e^{-2x} + C_2e^{4x} - \frac{2}{3}xe^{-2x}$]

Solution:

(Solve the associated homogeneous DE)

Auxiliary equation: $m^2 - 2m - 8 = 0 \Rightarrow (m+2)(m-4) = 0 \Rightarrow m = -2, 4$

Hence $y_C = C_1e^{-2x} + C_2e^{4x}$

(Next, we determine the form of y_p by using the non-homogeneous term on the right hand side together with all its derivatives; we need to multiply with the lowest power of x to eliminate duplication of functions in y_c)

$4\boxed{e^{-2x}} \xrightarrow{D} -8e^{-2x} \Rightarrow y_p = (Ae^{-2x})x = Axe^{-2x}$

(We complete this solution by finding the values of the parameter A in y_p using the method of undetermined coefficients)

$y_p = Axe^{-2x} \Rightarrow Dy = A[(1)(e^{-2x}) + (x)(-2e^{-2x})] = A(1-2x)e^{-2x}$

$\Rightarrow D^2y = A[(-2)(e^{-2x}) + (1-2x)(-2e^{-2x})] = A(4x-4)e^{-2x}$

$$D^2y - 2Dy - 8y = 4e^{-2x}$$

$$\Rightarrow A(4x-4)e^{-2x} - 2[A(1-2x)e^{-2x}] - 8[Axe^{-2x}] = 4e^{-2x}$$

$$\left\{\begin{matrix} xe^{-2x}: & 4A + 4A - 8A = 0 \\ e^{-2x}: & -4A - 2A = 4 \end{matrix}\right\} \Rightarrow A = -\frac{4}{6} = -\frac{2}{3}$$

Hence $y = y_c + y_p \Rightarrow \boldsymbol{y = C_1e^{-2x} + C_2e^{4x} - \frac{2}{3}xe^{-2x}}$

45. (Nice To Know) Without using the built in function keys for combinations, permutations, or factorials on your calculator, evaluate

(a) $P(2011,2)$ [Answer: 4042110]

Solution:

$$P(2011,2) = \frac{2011!}{(2011-2)!} = \frac{(2011)(2010)\cancel{2009!}}{\cancel{2009!}} = \mathbf{4042110}$$

(b) $C(n,3)$ [Answer: $\frac{n(n-1)(n-2)}{6}$]

Solution:

$$C(n,3) = \frac{n!}{(n-3)!\,3!} = \frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}(3)(2)(1)} = \boldsymbol{\frac{n(n-1)(n-2)}{6}}$$

46. (Nice To Know) Given n is a positive integer greater than 2, evaluate and simplify

$$\left|\frac{(n+1)!}{(n-1)4^{n+1}} \div \frac{n!}{(n-2)4^n}\right|.$$

[Answer: $\frac{(n+1)(n-2)}{4(n-1)}$]

Solution:

$$\left|\frac{(n+1)!}{(n-1)4^{n+1}} \div \frac{n!}{(n-2)4^n}\right| = \frac{(n+1)!\,(n-2)4^n}{(n-1)4^{n+1}n!} = \frac{(n+1)!}{n!} \times \frac{4^n}{4^{n+1}} \times \frac{n-2}{n-1}$$

$$= \frac{(n+1)\times\cancel{n!}}{\cancel{n!}} \times \frac{\cancel{4^n}}{\cancel{4^n}\times 4} \times \frac{(n-2)}{(n-1)} = \boldsymbol{\frac{(n+1)(n-2)}{4(n-1)}}$$

47. (Nice To Know) A shipment of 1000 items just arrived at a shop and a random sample of 10 is taken for inspection. The shipment is returned to the manufacturer if the sample contains 2 or more defectives. What is the probability of accepting the shipment if

it contains 25 defectives? (You can use the built in function keys for combinations, permutations, or factorials on your calculator.) [Answer: 0.9761]

Solution:

Probability of accepting the shipment

= Prob(sample has all 10 effectives) + Prob(sample has 9 effectives and 1 defective)

$$= \frac{C(975,10)}{C(1000,10)} + \frac{C(975,9) \times C(25,1)}{C(1000,10)} \approx 0.7754 + 0.2007 = \mathbf{0.9761}$$