Trigonometric Identities and Trigonometric Equations

Basic Trigonometric Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\csc\theta = \dfrac{1}{\sin\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

$\sin^2\theta + \cos^2\theta = 1$	$1 + \tan^2\theta = \sec^2\theta$	$1 + \cot^2\theta = \csc^2\theta$

(The last three identities are also called Pythagorean Identities.)

Example Prove the identity $\sin^2 x \tan^2 x = \tan^2 x - \sin^2 x$.

Solution:

(Pick the more complicated side to start with, either show that it can be rewritten into the other side, or it simplifies to an expression that the other side also can simplify to)

Right Hand Side (R.H.S.) $= \tan^2 x - \sin^2 x = \frac{\sin^2 x}{\cos^2 x} - \sin^2 x$

$= \sin^2 x \left(\frac{1}{\cos^2 x} - 1\right) = \sin^2 x (\sec^2 x - 1) = \sin^2 x (1 + \tan^2 x - 1)$

$= \sin^2 x \tan^2 x =$ Left Hand Side (L.H.S.)

Example For $\boldsymbol{u} = \left[e^t \cos\frac{3t}{4} - \frac{3}{4}e^t \sin\frac{3t}{4} \quad e^t \sin\frac{3t}{4} + \frac{3}{4}e^t \cos\frac{3t}{4} \quad 3e^t\right]$, evaluate and simplify $\|\boldsymbol{u}\|$.

Solution:

(Use the definition of norm)

$$\|\boldsymbol{u}\| = \sqrt{\left(e^t \cos\frac{3t}{4} - \frac{3}{4}e^t \sin\frac{3t}{4}\right)^2 + \left(e^t \sin\frac{3t}{4} + \frac{3}{4}e^t \cos\frac{3t}{4}\right)^2 + (3e^t)^2}$$

(Expand radicand using special product formula $(a \pm b)^2 = a^2 \pm 2ab + b^2$, and simplify by collecting like terms)

$$\left[e^{2t}\cos^2\left(\frac{3t}{4}\right) - \frac{3}{2}e^{2t}\sin\left(\frac{3t}{4}\right)\cos\left(\frac{3t}{4}\right) + \frac{9}{16}e^{2t}\sin^2\left(\frac{3t}{4}\right)\right] +$$

$$\left[e^{2t}\sin^2\left(\frac{3t}{4}\right) + \frac{3}{2}e^{2t}\sin\left(\frac{3t}{4}\right)\cos\left(\frac{3t}{4}\right) + \frac{9}{16}e^{2t}\cos^2\left(\frac{3t}{4}\right)\right] + 9e^{2t}$$

$$= \left(1 + \frac{9}{16}\right)e^{2t}\cos^2\left(\frac{3t}{4}\right) + \left(1 + \frac{9}{16}\right)e^{2t}\sin^2\left(\frac{3t}{4}\right) + 9e^{2t}$$

$$= \frac{25}{16}e^{2t}\cos^2\left(\frac{3t}{4}\right) + \frac{25}{16}e^{2t}\sin^2\left(\frac{3t}{4}\right) + 9e^{2t}$$

(Simplify using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$)

$$= \frac{25}{16}e^{2t}\left[\cos^2\left(\frac{3t}{4}\right) + \sin^2\left(\frac{3t}{4}\right)\right] + 9e^{2t} = \frac{25}{16}e^{2t}(1) + 9e^{2t} = \left(\frac{25}{16} + 9\right)e^{2t} = \frac{169}{16}e^{2t}$$

$$\Longrightarrow \|\boldsymbol{u}\| = \sqrt{\frac{169}{16}e^{2t}} = \frac{\sqrt{169}}{\sqrt{16}} \cdot \sqrt{e^{2t}} = \boldsymbol{\frac{13}{4}e^t}$$

Exercise

- Simplify
  - $\frac{\cos x}{1+\sin x} + \tan x$. [Answer: $\sec x$]
  - $2(r\cos\theta)^2 + 2(r\sin\theta)^2$. [Answer: $2r^2$]
  - $(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2 + (\rho\cos\varphi)^2$. [Answer: ρ^2]
  - Simplify $(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2 + 2(\rho\cos\varphi)^2$.

    [Answer: $\rho^2(1+\cos^2\varphi)$]

Sum and Difference Formulas

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Example Find the exact value of $\sin 15°$.

Solution:

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$$

Exercise

- Assume that $\sin\alpha = \frac{1}{3}$ and $\sin\beta = \frac{2}{3}$ and that α and β are between 0 and $\frac{\pi}{2}$, then evaluate $\sin(\alpha+\beta)$. [Answer: $\frac{\sqrt{5}+4\sqrt{2}}{9}$]
- Simplify $\sec\left(x - \frac{\pi}{2}\right)$. [Answer: $\csc x$]
- Evaluate $\sin\left(\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{5}{13}\right)$. [Answer: $\frac{5+12\sqrt{3}}{26}$]

Product Formulas

$$\sin u \sin v = \frac{\cos(u-v) - \cos(u+v)}{2}$$

$$\cos u \cos v = \frac{\cos(u+v) + \cos(u-v)}{2}$$

$$\sin u \cos v = \frac{\sin(u+v) + \sin(u-v)}{2}$$

Double-Angle Formulas

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Exercise Prove the identity $\frac{\sin(2x)}{\sin x} - \frac{\cos(2x)}{\cos x} = \sec x$.

Power Reducing Formulas

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

Half-Angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$$

Exercise Find the exact value of $\tan\frac{\pi}{8}$. [Answer: $\sqrt{2}-1$]

Solving Trigonometric Equations

Example Solve $2\cos^2\theta - \cos\theta = 1$ for θ satisfying $0 \le \theta < 2\pi$.

Solution:

(Look at the equation as a quadratic equation in $\cos\theta$; solve for $\cos\theta$ and then solve for θ)

$2\cos^2\theta - \cos\theta - 1 = 0$

$D = b^2 - 4ac = (-1)^2 - 4(2)(-1) = 1 + 8 = 9 = 3^2$

(Since the discriminant D is a perfect square, we can factor the quadratic expression using the *ac*-method)

$$2\cos^2\theta - 2\cos\theta + \cos\theta - 1 = 0 \Rightarrow 2\cos\theta(\cos\theta - 1) + (\cos\theta - 1) = 0$$

$\Rightarrow (\cos\theta - 1)(2\cos\theta + 1) = 0 \Rightarrow \cos\theta = 1 \quad \text{or} \quad \cos\theta = -\frac{1}{2}$

$\cos\theta = 1 \Rightarrow \boldsymbol{\theta = 0}$;

$\cos\theta = -\frac{1}{2} \Rightarrow$ Reference angle ϕ satisfies $\cos\phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$

Since $\cos\theta < 0$, θ has terminal side in quadrant 2 or quadrant 3 and hence

$\theta = \pi - \phi$ or $\pi + \phi \Rightarrow \boldsymbol{\theta = \frac{2\pi}{3}}$ or $\boldsymbol{\theta = \frac{4\pi}{3}}$

Exercise

- Solve $\sin 3\theta = 0$ for θ satisfying $0 \le \theta < 2\pi$. [Answer: $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$]
- Solve $\sin 2\theta = \cos\theta$ for θ satisfying $0 \le \theta < 2\pi$. [Answer: $\theta = \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$]
- Solve $3\cos\theta = 5\sin\theta$ for θ satisfying $0 \le \theta < 2\pi$.

  [Answer: $\theta = \tan^{-1}\frac{3}{5}, \pi + \tan^{-1}\frac{3}{5} \approx 0.540, 3.682$]
- Solve $\sin(2\theta) - \sin\theta = 0$ for θ satisfying $0 \le \theta < 2\pi$. [Answer: $\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$]
- Solve $\sin^2\theta + \cos\theta - \cos^2\theta = 0$ for θ satisfying $0 \le \theta < 2\pi$. [Answer: $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$]
- Solve $\sin x + \cos x = 1$ in $[0, 2\pi)$. [Answer: $x = 0, \frac{\pi}{2}$]

Example Find the third positive answer for $\sin\left(2t+\frac{\pi}{3}\right)=\frac{1}{2}$.

Solution:

(First find the general solution, that is all solutions, of the equation; recall that according to the CAST diagram, $\sin\theta=\frac{1}{2}$ has solution for θ terminating in quadrant 1 and quadrant 2, with reference angle $\frac{\pi}{6}$)

$$\sin\left(2t+\frac{\pi}{3}\right)=\frac{1}{2}\Rightarrow 2t+\frac{\pi}{3}=\frac{\pi}{6}+2n\pi \text{ or } \frac{5\pi}{6}+2n\pi$$

$$\Rightarrow 2t=\frac{\pi}{6}-\frac{\pi}{3}+2n\pi \text{ or } \frac{5\pi}{6}-\frac{\pi}{3}+2n\pi$$

$$\Rightarrow 2t=-\frac{\pi}{6}+2n\pi \text{ or } \frac{3\pi}{6}+2n\pi \Rightarrow t=-\frac{\pi}{12}+n\pi \text{ or } \frac{\pi}{4}+n\pi$$

(Determine the values of n so that t is positive)

$$-\frac{\pi}{12}+n\pi>0\Rightarrow n\pi>\frac{\pi}{12}\Rightarrow n>\frac{1}{12}$$

$$\Rightarrow t=\frac{11}{12}\pi \text{ (with } n=1),\frac{23}{12}\pi \text{ (with } n=2),\cdots$$

$$\frac{\pi}{4}+n\pi>0\Longrightarrow n\pi>-\frac{\pi}{4}\Longrightarrow n>-\frac{1}{4}$$

$$\Rightarrow t=\frac{\pi}{4}\ \text{ (with } n=0),\frac{5}{4}\pi \text{ (with } n=1),\cdots$$

Since $0<\frac{1}{4}\pi<\frac{11}{12}\pi<\frac{5}{4}\pi<\frac{23}{12}\pi$, the third positive solution is $\boldsymbol{\frac{5}{4}\pi}$.

Example Given $A>0$ and $0\le\varphi<2\pi$, solve $A\sin\varphi=0.2$ and $A\cos\varphi=-0.3$ for A and φ.

Solution:

(To find A, we eliminate φ from the two given equations by using a trigonometric identity and solve for A) $\quad (A\sin\varphi)^2+(A\cos\varphi)^2=(0.2)^2+(-0.3)^2$

$$\Rightarrow A^2\sin^2\varphi+A^2\cos^2\varphi=0.04+0.09\Rightarrow A^2(\sin^2\varphi+\cos^2\varphi)=0.13$$

$$\Rightarrow A^2(1)=0.13\Rightarrow A^2=0.13\Longrightarrow A=\sqrt{0.13}\Longrightarrow \boldsymbol{A\approx 0.361}$$

(Divide one of the equations in the original system by the other to eliminate A): $\frac{A\sin\varphi}{A\cos\varphi}=\frac{0.2}{-0.3}\Rightarrow \tan\varphi=-\frac{2}{3}$

(Find ϕ, the reference angle of φ) $\quad \phi=\tan^{-1}\left(\left|-\frac{2}{3}\right|\right)=\tan^{-1}\left(\frac{2}{3}\right)$

(Since $\sin\varphi=\frac{0.2}{A}>0$ and $\cos\varphi=\frac{-0.3}{A}<0$, the angle φ terminates in the second quadrant and hence $\varphi=\pi-\phi$) $\quad \varphi=\pi-\phi=\pi-\tan^{-1}\left(\frac{2}{3}\right)\approx \mathbf{2.554\ (radians)}$

Solving a Triangle

$$\text{Law of Sines: } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\text{Law of Cosines: } \begin{cases} a^2 = b^2 + c^2 - 2bc\cos A \\ b^2 = a^2 + c^2 - 2ac\cos B \\ c^2 = a^2 + b^2 - 2ab\cos C \end{cases}$$

(Possible Ambiguous case in solving a triangle given two sides and the angle opposite one of them)

Exercise Solve the triangle

- ΔABC with $b = 15, c = 20$ and $B = 29°$.

  [Answer: Two solutions: $A \approx 111°, C \approx 40°, a \approx 29$ or $A \approx 11°, C \approx 140°, a \approx 6$]
- ΔXYZ with $x = 23.5, y = 9.8$ and $X = 39.7°$.

  [Answer $Y \approx 15.4°, Z \approx 124.9°, a \approx 30.2$]
- ΔXYZ with $x = 4.56, X = 43°$ and $Z = 57°$. [Answer: $Y = 80°, y \approx 6.58, z \approx 5.61$]
- ΔABC with $a = 3.5, b = 4.7$ and $c = 2.8$.

  [Answer: $A \approx 47.80°, B \approx 95.86°, C \approx 36.34°$]
- ΔXYZ with $x = 32, z = 48$ and $Y = 125.2°$. [Answer $X \approx 22.0°, Z \approx 32.8°, y \approx 71$]