Polynomials

<u>Fundamental Operations with Polynomials</u>

Evaluating Polynomial Functions

Addition, Subtraction and Multiplication of Polynomials

Multiplying Binomials:

- Square of a sum: $(a+b)^2 = a^2 + 2ab + b^2$
- Square of a difference: $(a-b)^2 = a^2 - 2ab + b^2$
- Product of a sum and a difference: $(a+b)(a-b) = a^2 - b^2$
- the FOIL Method: $(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$
- Cube of a sum: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- Cube of a difference: $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

<u>Example</u> Expand $(1+x)\left(1+2x-2x^2+\frac{4}{3}x^3\right)$ and simplify by collecting like terms.

Solution:

(Systematically multiply each term in the first factor into each factor in the second factor)

$(1)(1)+(1)(2x)+(1)(-2x^2)+(1)\left(\frac{4}{3}x^3\right)+(x)(1)+(x)(2x)+(x)(-2x^2)+$

$(x)\left(\frac{4}{3}x^3\right) = 1+2x-2x^2+\frac{4}{3}x^3+x+2x^2-2x^3+\frac{4}{3}x^4$

(Simplify by collecting like terms)

$= 1+(2x+x)+(-2x^2+2x^2)+\left(\frac{4}{3}x^3-2x^3\right)+\frac{4}{3}x^4 = \mathbf{1+3x-\frac{2}{3}x^3+\frac{4}{3}x^4}$

<u>Exercise</u> Perform the multiplication.

- $(2x-7)(3x+4)$ [Answer: $6x^2-13x-28$]
- $(3y^4-5)^2$ [Answer: $9y^8-30y^4+25$]
- $(y^2+5x)(y^2-5x)$ [Answer: y^4-25x^2]
- $(4x^4y-7x^2y+3y)(2y-3x^2y)$ [Answer: $29x^4y^2-12x^6y^2-23x^2y^2+6y^2$]

Division of Polynomials: Ordinary (Long Division)

- dividend = (divisor)(quotient) + (remainder)
- $\dfrac{\text{dividend}}{\text{divisor}} = \text{quotient} + \dfrac{\text{remainder}}{\text{divisor}}$

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Example Perform long division $\frac{2x^2+1}{x-3}$ [Answer: $\frac{2x^2+1}{x-3} = 2x + 6 + \frac{19}{x-3}$]

Solution:

(Use long division to find the quotient and the remainder when the dividend $2x^2 + 1$ is divided by the divisor $x - 3$, then we can write $\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$)

$$
\begin{array}{rrrrrl}
 & & & 2x & +6 & = \text{Quotient} \\
x \quad -3 & & 2x^2 & +0x & +1 & \\
 & -) & 2x^2 & -6x & & \\
 & & & 6x & +1 & \\
 & & -) & 6x & -18 & \\
 & & & & 19 & = \text{Remainder}
\end{array}
$$

$$\mathbf{\frac{2x^2+1}{x-3} = 2x + 6 + \frac{19}{x-3}}$$

Exercise Perform the following long division

- $\frac{x^2-5}{x-2}$ [Answer: $\frac{x^2-5}{x-2} = x + 2 - \frac{1}{x-2}$]
- $\frac{x^4-81}{x^2+3x-1}$ [Answer: $\frac{x^4-81}{x^2+3x-1} = x^2 - 3x + 10 + \frac{-33x-71}{x^2+3x-1}$]
- $\frac{15x^3+101x^2+384x+24}{(5x-1)(x^2+6x+25)}$ [Answer: $3 + \frac{14x^2+27x+99}{(5x-1)(x^2+6x+25)}$]

Factoring Polynomials

- Factoring out the Greatest Common Factor: $ax + ay = a(x + y)$

  Exercise Factor completely $12x^2y^3 - 20x^3y$. [Answer: $4x^2y(3y^2 - 5x)$]
- Factoring Difference of Two Square: $a^2 - b^2 = (a + b)(a - b)$

  Exercise Factor completely $6x^4 - 96y^4$. [Answer: $6(x^2 + 4y^2)(x + 2y)(x - 2y)$]
- Factoring a Difference or Sum of Two Cubes:

  $$a^3 - b^3 = (a - b)(a^2 + ab + b^2); \qquad a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

  Exercise Factor completely $24t^3 - 375$. [Answer: $3(2t - 5)(4t^2 + 10t + 25)$]
- Factoring Perfect Square Trinomials:

  $$a^2 + 2ab + b^2 = (a + b)^2; \qquad a^2 - 2ab + b^2 = (a - b)^2$$

  Exercise Factor completely $25y^2 + 3yy + 9$. [Answer: $(5y + 3)^2$]

Polynomials

- Factoring $ax^2 + bx + c$: Trial and Error, or the *ac*-Method:
  - Factor out the largest common factor
  - Multiply the leading coefficient *a* and the constant *c*
  - Try to factor the product *ac* so that the sum of the factors is *b* (that is, find integers *p* and *q* such that $pq = ac$ and $p + q = b$)
  - Split the middle term (that is, write it as a sum of the factors found in the previous step)
  - Factor by grouping

  Example Factor completely $3t^2 + 10t - 8$.

  Solution:

  (Find *p* and *q* in the *ac*-method) Use trial and error to find *p* and *q* satisfying

  $$\begin{cases} pq = (3)(-8) = -24 \\ p + q = 10 \end{cases} \Rightarrow \quad p \text{ and } q \text{ are } 12 \text{ and } -2$$

  (Split the middle term using the previous result and factor by grouping)

  $$3t^2 + 10t - 8 = 3t^2 + 12t - 2t - 8 = (3t^2 + 12t) + (-2t - 8)$$
  $$= 3t(t + 4) - 2(t + 4) = \mathbf{(t + 4)(3t - 2)}$$

  Exercise Factor completely

  - $2y^3 - 14y^2 + 24y$. [Answer: $2y(y - 3)(y - 4)$]

- Factoring Polynomials with Four Terms: try factoring by grouping

  Example Factor completely $x^3 - 3x^2 - 4x + 12$.

  Solution:

  $$x^3 - 3x^2 - 4x + 12 = (x^3 - 3x^2) + (-4x + 12) = x^2(x - 3) - 4(x - 3)$$
  $$= (x - 3)(x^2 - 4) = (x - 3)(x^2 - 2^2) = \mathbf{(x - 3)(x + 2)(x - 2)}$$

- Factoring by Substitution

Example Simplify $\left[\frac{k(k+1)}{2}\right]^2 + (k + 1)^3$.

Solution:

(Factor out common factor before expanding and simplifying)

$$\left[\frac{k(k + 1)}{2}\right]^2 + (k + 1)^3 = \frac{k^2(k + 1)^2}{2^2} + (k + 1)^3 = (k + 1)^2\left[\frac{1}{4}k^2 + (k + 1)\right]$$

$$= (k + 1)^2\left[\frac{k^2+4k+4}{4}\right] = \frac{1}{4}(k + 1)^2(k^2 + 4k + 4) = \mathbf{\frac{1}{4}(k + 1)^2(k + 2)^2}$$

(Factor $k^2 + 4k + 4$ using the special product formula $a^2 + 2ab + b^2 = (a + b)^2$)

$= \frac{1}{4}(\boldsymbol{k} + \mathbf{1})^2(\boldsymbol{k} + \mathbf{2})^2$

Exercise Determine if $(x + 3)$ is a factor of $x^3 + 2x^2 - 5x - 6$.

[Answer: yes, $x^3 + 2x^2 - 5x - 6 = (x + 3)(x^2 - x - 2)$]

Solving Equations by Factoring

- Zero Factor Property: $a \cdot b = 0 \Leftrightarrow a = 0$ or $b = 0$
- Pythagorean Theorem: For a right triangle with legs a, b and hypotenuse c, $a^2 + b^2 = c^2$

Example Solve $21x^2 \geq 8 - 2x$.

Solution:

(Replace the inequality sign by the equal sign, and rewrite the resulting equation to have a zero on one side) $21x^2 = 8 - 2x \Longrightarrow 21x^2 + 2x - 8 = 0$

(Solve the equation by factoring completely the nonzero side; in this example, use the ac-method) $\begin{cases} pq = (21)(-8) = -168 \\ p + q = 2 \end{cases} \Rightarrow$ p and q are 14 and -12

$\Rightarrow 21x^2 + 14x - 12x - 8 = 0 \Longrightarrow (21x^2 + 14x) + (-12x - 8) = 0$

$\Rightarrow 7x(3x + 2) - 4(3x + 2) = 0 \Longrightarrow (3x + 2)(7x - 4) = 0$

$\Rightarrow 3x+2=0$ or $7x - 4 = 0 \Rightarrow x = -\frac{2}{3}$ or $x = \frac{4}{7}$

(Separate the real number line $(-\infty, \infty)$ into intervals by using the solution(s) of the equation; decide which of the intervals created belong to the solution set by using a test number – pick a convenient number in each of the intervals and check whether the original inequality is satisfied by that number; if it does, the interval that contains this number is part of the solution set)

Interval	Test Number x	$21x^2 \geq 8 - 2x$?
$\left(-\infty, -\frac{2}{3}\right)$	-1	$21(-1)^2 \geq 8 - 2(-1)$? $\Leftrightarrow$ $21 \geq 8 + 2$? *Yes*!
$\left(-\frac{2}{3}, \frac{4}{7}\right)$	0	$21(0)^2 \geq 8 - 2(0)$? $\Leftrightarrow$ $0 \geq 8 - 0$? *No*!
$\left(\frac{4}{7}, \infty\right)$	1	$21(1)^2 \geq 8 - 2(1)$? $\Leftrightarrow$ $21 \geq 8 - 2$? *Yes*!

(Determine whether each of the numbers used to create the intervals satisfies the inequality and hence should be included in the solution set)

$x=-\frac{2}{3}: 21\left(-\frac{2}{3}\right)^2 \geq 8-2\left(-\frac{2}{3}\right)\ ? \quad \Leftrightarrow \quad \frac{28}{3} \geq 8+\frac{4}{3}\ ? \quad \Leftrightarrow \quad \frac{28}{3} \geq \frac{28}{3}\ ? \quad Yes!$

$x=\frac{4}{7}: 21\left(\frac{4}{7}\right)^2 \geq 8-2\left(\frac{4}{7}\right)\ ? \quad \Leftrightarrow \quad \frac{48}{7} \geq 8-\frac{8}{7}\ ? \quad \Leftrightarrow \quad \frac{48}{7} \geq \frac{48}{7}\ ? \quad Yes!$

Hence the solution set is $\left(-\infty,-\frac{2}{3}\right] \cup \left[\frac{4}{7}, \infty\right)$.

Exercise

- Solve the equation $x^3-2x^2-9x+18=0$. [Answer: $x=-3, 2, 3$]
- Solve $x^3-4x<0$. [Answer: $(-\infty,-2) \cup (0,2)$]