Matrices

Definitions and Basic Operation

1. Two matrices are equal if and only if they have the same dimension, and the elements must respectively be equal
2. Matrix addition and subtraction
3. Scalar multiplication
4. Commutative Law: A + B = B + A
5. Associative Law: A + (B + C) = (A + B) + C
6. k(A + B) = kA + kB
7. A + 0 = A

Example Evaluate $-2\begin{bmatrix}-4 & 7\\ -2 & 8\end{bmatrix} + 3\begin{bmatrix}3 & -6\\ 7 & -8\end{bmatrix}$.

Solution:

$$-2\begin{bmatrix}-4 & 7\\ -2 & 8\end{bmatrix} + 3\begin{bmatrix}3 & -6\\ 7 & -8\end{bmatrix} = \begin{bmatrix}(-2)(-4) & (-2)(7)\\ (-2)(-2) & (-2)(8)\end{bmatrix} + \begin{bmatrix}(3)(3) & (3)(-6)\\ (3)(7) & (3)(-8)\end{bmatrix}$$

$$= \begin{bmatrix}8 & -14\\ 4 & -16\end{bmatrix} + \begin{bmatrix}9 & -18\\ 21 & -24\end{bmatrix} = \begin{bmatrix}8+9 & (-14)+(-18)\\ 4+21 & (-16)+(-24)\end{bmatrix} = \begin{bmatrix}\mathbf{17} & \mathbf{-32}\\ \mathbf{25} & \mathbf{-40}\end{bmatrix}$$

Exercise Evaluate $\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} + 2\begin{bmatrix}1 & -1\\ 4 & 1\end{bmatrix}$ [Answer: $\begin{bmatrix}4 & -2\\ 8 & 4\end{bmatrix}$]

Multiplication

1. Compatibility (multiply row into column)
2. Matrix multiplication is not commutative
3. Distributive property: A(B + C) = AB + AC
4. Identity Matrix: AI = IA = A
5. Inverse Matrix: $AA^{-1} = A^{-1}A = I$

Example Evaluate $\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Solution:

(The product of a 3×2 matrix and a 2×1 matrix is a 3×1 matrix)

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} (1)(1)+(1)(-2) \\ (2)(1)+(1)(-2) \\ (3)(1)+(4)(-2) \end{bmatrix} = \begin{bmatrix} 1-2 \\ 2-2 \\ 3-8 \end{bmatrix} = \begin{bmatrix} \mathbf{-1} \\ \mathbf{0} \\ \mathbf{-5} \end{bmatrix}$$

Exercise Evaluate the following

- $\begin{bmatrix} 4 & 3 \end{bmatrix}\begin{bmatrix} 100 & 80 & 120 \\ 160 & 120 & 100 \end{bmatrix}$ [Answer: $\begin{bmatrix} 880 & 680 & 780 \end{bmatrix}$]
- $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ [Answer: $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$]
- $\begin{bmatrix} 2 & 4 & 5 \\ 1 & 3 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}$ [Answer: $\begin{bmatrix} 128 \\ 130 \end{bmatrix}$]
- $\begin{bmatrix} 9 & 9 & -2 \\ 9 & -2 & 8 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ -8 & 6 \\ -4 & 4 \end{bmatrix}$ [Answer: $\begin{bmatrix} -28 & 118 \\ 20 & 92 \end{bmatrix}$]
- $\begin{bmatrix} 2 & 4 & 5 \\ 1 & 3 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 5 & 6 \end{bmatrix}$ [Answer: $\begin{bmatrix} 45 & 44 \\ 44 & 46 \end{bmatrix}$]
- $\begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 4 & -6 \\ 1 & 2 \end{bmatrix}$ [Answer: $\begin{bmatrix} 10 & 6 \\ 7 & -28 \\ -4 & 20 \end{bmatrix}$]
- $\begin{bmatrix} 1 & 6 \\ 3 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & -3 \end{bmatrix}$ [Answer: $\begin{bmatrix} 15 & 1 & 17 \\ -1 & 3 & -18 \\ 2 & -2 & 14 \end{bmatrix}$]
- $\begin{bmatrix} 2 & 8 & -7 \\ 0 & -4 & 4 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$ [Answer: $\begin{bmatrix} 12 \\ -8 \\ 8 \end{bmatrix}$]
- $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ [Answer: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$]
- $3\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ [Answer: $\begin{bmatrix} 2 & -1 \\ 0 & 11 \end{bmatrix}$]

Matrices

Determinants

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$; $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = aek + bfg + cdh - ceg - afh - bdk$

Example Evaluate the determinant of $\begin{bmatrix} \sqrt{2} & -3 \\ -4 & -\sqrt{2} \end{bmatrix}$.

Solution:

$$\begin{vmatrix} \sqrt{2} & -3 \\ -4 & -\sqrt{2} \end{vmatrix} = (\sqrt{2})(-\sqrt{2}) - (-3)(-4) = -2 - 12 = \mathbf{-14}$$

Example Evaluate the determinant of $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 4 & 1 \end{bmatrix}$.

Solution:

$$\begin{vmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 4 & 1 \end{vmatrix}$$

$$= (0)(0)(1) + (2)(1)(0) + (1)(1)(4) - (1)(0)(0) - (0)(1)(4) - (2)(1)(1)$$

$$= 0 + 0 + 4 - 0 - 0 - 2 = \mathbf{2}$$

Exercise Evaluate the determinant of the matrix.

- $\begin{bmatrix} 2 & 5 \\ 5 & -2 \end{bmatrix}$. [Answer: -29]
- $\begin{bmatrix} 13 & -3 & 7 \\ 1 & 1 & 1 \\ 4 & -2 & 3 \end{bmatrix}$. [Answer: 20]

Inverse

Shortcut in finding the inverse of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\text{ad}-\text{bc}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc) & a/(ad-bc) \end{bmatrix}$$

Example Evaluate the inverse of $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{(1)(3)-(-1)(2)} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{3/5} & \mathbf{1/5} \\ \mathbf{-2/5} & \mathbf{1/5} \end{bmatrix}$$

(Check that $AA^{-1} = I$)

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix} = \begin{bmatrix} (1)(3/5)+(-1)(-2/5) & (1)(1/5)+(-1)(1/5) \\ (2)(3/5)+(3)(-2/5) & (2)(1/5)+(3)(1/5) \end{bmatrix}$$

$$= \begin{bmatrix} (3/5)+(2/5) & (1/5)+(-1/5) \\ (6/5)+(-6/5) & (2/5)+(31/5) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise Evaluate the inverse of the matrix

- $\begin{bmatrix} 0 & -1/5 \\ 1/2 & 4/5 \end{bmatrix}^{-1}$ [Answer: $\begin{bmatrix} 8 & 2 \\ -5 & 0 \end{bmatrix}$]
- $\begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}^{-1}$ [Answer: $\begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$]
- $\begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$ [Answer: $\begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$]

Solving Systems of Linear Equations

- using inverse matrix: $AX = C \Leftrightarrow X = A^{-1}C$
- Cramer's rule:
  - $\left\{\begin{matrix} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{matrix}\right\} \Rightarrow x = \dfrac{D_x}{D} = \dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},\quad y = \dfrac{D_y}{D} = \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$
  - $\left\{\begin{matrix} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{matrix}\right\} \Rightarrow x = \dfrac{D_x}{D} = \dfrac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\quad y = \dfrac{D_y}{D} = \dfrac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$

$$z = \frac{D_z}{D} = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Example Use Cramer's rule to solve the system $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$.

Solution:

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix}} = \frac{(7)(-2)-(5)(-3)}{(2)(-2)-(5)(5)} = \frac{-14+15}{-4-25} = \frac{1}{-29} = -\frac{1}{29}$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix}} = \frac{(2)(-3)-(7)(5)}{(2)(-2)-(5)(5)} = \frac{-6-35}{-4-25} = \frac{-41}{-29} = \frac{41}{29}$$

Hence $(\boldsymbol{x}, \boldsymbol{y}) = \left(-\frac{\mathbf{1}}{\mathbf{29}}, \frac{\mathbf{41}}{\mathbf{29}}\right)$.

Example Use Cramer's rule 3t5o solve the system $\begin{cases} x - 3y + 7z = 13 \\ x + y + z = 1 \\ x - 2y + 3z = 4 \end{cases}$.

Solution:

$$D = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$=(1)(1)(3) + (-3)(1)(1) + (7)(1)(-2) - (7)(1)(1) - (1)(1)(-2) - (-3)(1)(3)$$

$$= 3 - 3 - 14 - 7 + 2 + 9 = -10$$

$$D_x = \begin{vmatrix} 13 & -3 & 7 \\ 1 & 1 & 1 \\ 4 & -2 & 3 \end{vmatrix}$$

$$= (13)(1)(3) + (-3)(1)(4) + (7)(1)(-2) - (7)(1)(4) - (13)(1)(-2) - (-3)(1)(3)$$

$$= 39 - 12 - 14 - 28 + 26 + 9 = 20$$

$$\Longrightarrow x = \frac{D_x}{D} = \frac{20}{-10} = -2$$

$$D_y = \begin{vmatrix} 1 & 13 & 7 \\ 1 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix}$$

$$=(1)(1)(3) + (13)(1)(1) + (7)(1)(4) - (7)(1)(1) - (1)(1)(4) - (13)(1)(3)$$

$$= 3 + 13 + 28 - 7 - 4 - 39 = -6$$

$$\Longrightarrow y = \frac{D_y}{D} = \frac{-6}{-10} = \frac{3}{5}$$

$$D_z = \begin{vmatrix} 1 & -3 & 13 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$=(1)(1)(4) + (-3)(1)(1) + (13)(1)(-2) - (13)(1)(1) - (1)(1)(-2) - (-3)(1)(4)$$

$$= 4 - 3 - 26 - 13 + 2 + 12 = -24$$

$\Longrightarrow z = \frac{D_z}{D} = \frac{-24}{-10} = \frac{12}{5}$

Hence $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \left(\boldsymbol{-2}, \frac{\boldsymbol{3}}{\boldsymbol{5}}, \frac{\boldsymbol{12}}{\boldsymbol{5}}\right)$.

Exercise Use Cramer's rule to solve the system.

- $\begin{Bmatrix} 2x & - & y & = & 5 \\ x & - & 2y & = & 1 \end{Bmatrix}$ [Answer: $(x, y) = (3,1)$]
- $\begin{Bmatrix} 2x & + & 9y & = & -2 \\ 4x & - & 3y & = & 3 \end{Bmatrix}$ [Answer: $(x, y) = \left(\frac{1}{2}, -\frac{1}{3}\right)$]
- $\begin{Bmatrix} 2x & + & 5y & = & 7 \\ 3x & - & 2y & = & 1 \end{Bmatrix}$ [Answer: $(x, y) = (1,1)$]
- $\begin{Bmatrix} 3x & + & 5y & - & z & = & -2 \\ x & - & 4y & + & 2z & = & 13 \\ 2x & + & 4y & + & 3z & = & 1 \end{Bmatrix}$ [Answer: $(x, y, z) = (3, -2, 1)$]
- $\begin{Bmatrix} x & - & 3y & - & 7z & = & 6 \\ 2x & + & 3y & + & z & = & 9 \\ 4x & + & y & & & = & 7 \end{Bmatrix}$ [Answer: $(x, y, z) = (1, 3, -2)$]
- $\begin{Bmatrix} & & 6y & + & 6z & = & -1 \\ 8x & & & + & 6z & = & -1 \\ 4x & + & 9y & & & = & 8 \end{Bmatrix}$ [Answer: $(x, y, z) = \left(\frac{1}{2}, \frac{2}{3}, -\frac{5}{6}\right)$]