Limits

Example Evaluate $\lim_{x\to 2} \frac{\sqrt{4x^2+9}}{x+1}$.

Solution:

(The limit of the quotient is the quotient of the limits, provided the limit of the denominator is non-zero; moreover, both the radical function in the numerator and the polynomial in the denominator are continuous for all values of x, hence these two limits can be evaluated by direct substitution.)

$$\lim_{x\to 2} \frac{\sqrt{4x^2+9}}{x+1} = \frac{\lim_{x\to 2}\sqrt{4x^2+9}}{\lim_{x\to 2}(x+1)} = \frac{\sqrt{4(2^2)+9}}{2+1} = \frac{\sqrt{16+9}}{3} = \frac{\sqrt{25}}{3} = \mathbf{\frac{5}{3}}$$

Example Evaluate $\lim_{x\to 1} \frac{x^2-x}{x^2-1}$.

Solution:

(Note that the limits for both the numerator and the denominator are zeros, so direct substitution doesn't work, instead, we factor both the numerator and the denominator and cancel common factor before trying direct substitution)

$$\lim_{x\to 1} \frac{x^2-x}{x^2-1} = \lim_{x\to 1} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x\to 1} \frac{x\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \lim_{x\to 1} \frac{x}{(x+1)} = \frac{\lim_{x\to 1} x}{\lim_{x\to 1}(x+1)} = \mathbf{\frac{1}{2}}$$

Exercise Evaluate the limit.

- $\lim_{x\to 0} \cos(3x^2)$ [Answer: 1]
- $\lim_{x\to 2} 2x\cos(x^3-8)$ [Answer: 4]
- $\lim_{y\to 0} \frac{ay^6}{a^3y^6+y^7}$ $\left[\text{Answer: } \frac{1}{a^2}\right]$
- $\lim_{a\to 2^-} \frac{-1}{a-2}$ [Answer: $+\infty$]
- $\lim_{x\to 1^+} [(\ln x)^2 + 2\ln x]$ [Answer: 0]

Example Evaluate $\lim_{x\to\infty} \frac{5^x-2}{5^x}$.

Solution:

(As x approaches infinity, both the denominator 5^x and the numerator 5^x-2 approach infinity, so direct substitution won`t work; instead, we can split the numerator and write the expression as the difference of two terms, and we get the answer by subtracting the limit of the second term from that of the first term)

$$\lim_{x\to\infty} \frac{5^x-2}{5^x} = \lim_{x\to\infty}\left(\frac{5^x}{5^x} - \frac{2}{5^x}\right) = \lim_{x\to\infty}\left(1 - \frac{2}{5^x}\right)$$

$= \lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{2}{5^x} = 1 - 0 = \mathbf{1}$

(The limit of the first term, which is a constant, remains the same constant; the limit of the second term is 0, since the limit of the numerator is 2 and the limit of the denominator is infinity.)

Example Evaluate $\lim_{n\to\infty} \frac{3n^3+2n^2}{n^3-4}$.

Solution:

(To find the limit of a rational function – the quotient of two polynomials, we can divide each term in the numerator and the denominator by the highest power of the variable, in this case n^3, before finding the limit for each of the results)

$$\lim_{n\to\infty} \frac{3n^3+2n^2}{n^3-4} = \lim_{n\to\infty} \frac{(3n^3/n^3)+(2n^2/n^3)}{(n^3/n^3)-(4/n^3)} = \lim_{n\to\infty} \frac{(3)+(2/n)}{(1)-(4/n^3)}$$

$$= \frac{\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{2}{n}}{\lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{4}{n^3}} = \frac{3+0}{1-0} = \frac{3}{1} = \mathbf{3}$$

Example Evaluate $\lim_{n\to\infty} \frac{[(n+1)!]^2(3n)!}{(n!)^2[3(n+1)]!}$

Solution:

$$\lim_{n\to\infty} \frac{[(n+1)!]^2(3n)!}{(n!)^2[3(n+1)]!} = \lim_{n\to\infty} \left\{\frac{[(n+1)!]^2}{(n!)^2} \times \frac{(3n)!}{[3(n+1)]!}\right\}$$

$$= \lim_{n\to\infty} \left\{\left[\frac{(n+1)!}{n!}\right]^2 \times \frac{(3n)!}{[3n+3]!}\right\} = \lim_{n\to\infty} \left\{\left[\frac{(n+1)\times n!}{n!}\right]^2 \times \frac{(3n)!}{(3n+3)(3n+2)(3n+1)\times[3n]!}\right\}$$

$$= \lim_{n\to\infty} \left\{(n+1)^2 \times \frac{1}{(3n+3)(3n+2)(3n+1)}\right\} = \lim_{n\to\infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)}$$

(Since the numerator is a polynomial of degree two whereas the denominator is a polynomial of degree three, as *n* approaches infinity, the denominator grows much faster than the numerator and hence the answer for the limit is **0**)

Exercise Evaluate the limit.

- $\lim_{x\to\infty} \frac{1-4x^2}{x+2x^2}$ [Answer: -2]
- $\lim_{n\to\infty} \frac{n}{n+1}$ [Answer: 1]
- $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1}$ [Answer: 0]
- $\lim_{x\to\infty} \frac{6}{1+(2/x)}$ [Answer: 6]

- $\lim_{b\to\infty} \tan^{-1} b$ $\left[\text{Answer: } \frac{\pi}{2}\right]$
- $\lim_{x\to\infty} \tan^{-1}\left(\frac{2x-3}{5}\right)$ $\left[\text{Answer: } \frac{\pi}{2}\right]$
- $\lim_{x\to\infty} \sec^2\left(\frac{1}{2x^2}\right)$ [Answer: 1]
- $\lim_{t\to-\infty} 2\sqrt{3-t}$ [Answer: $+\infty$]