Integration Methods

Basic Formulae on Integration

• $\int c \cdot f(u)\, du = c \int f(u)\, du$	• $\int e^u\, du = e^u + C$
• $\int [f(u) + g(u)]\, du = \int f(u)\, du + \int g(u)\, du$	
• $\int u^n\, du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$	• $\int \frac{du}{u} = \ln\lvert u\rvert + C$
• $\int \sin u\, du = -\cos u + C$	• $\int \cos u\, du = \sin u + C$
• $\int \tan u\, du = -\ln\lvert\cos u\rvert + C$	• $\int \cot u\, du = \ln\lvert\sin u\rvert + C$
• $\int \sec u\, du = \ln\lvert\sec u + \tan u\rvert + C$	• $\int \csc u\, du = \ln\lvert\csc u - \cot u\rvert + C$
• $\int \sec^2 u\, du = \tan u + C$	• $\int \csc^2 u\, du = -\cot u + C$
• $\int \sec u \tan u\, du = \sec u + C$	• $\int \csc u \cot u\, du = -\csc u + C$
• $y_{rms} = \sqrt{\frac{1}{T}\int_0^T y^2\, dx}$	
• Definite Integral: $\int_a^b f(x)\, dx = F(b) - F(a)$	

Example Evaluate and simplify $\int_{\pi/6}^{\pi/4} \frac{1+\cos x}{\sin x}\, dx$.

Solution:

$$\int_{\pi/6}^{\pi/4} \frac{1+\cos x}{\sin x}\, dx = \int_{\pi/6}^{\pi/4} \left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right) dx$$

$$= \int_{\pi/6}^{\pi/4} (\csc x + \cot x)\, dx = \ln\lvert\csc x - \cot x\rvert + \ln\lvert\sin x\rvert\Big|_{\pi/6}^{\pi/4}$$

$$= \left(\ln\left|\csc\frac{\pi}{4} - \cot\frac{\pi}{4}\right| + \ln\left|\sin\frac{\pi}{4}\right|\right) - \left(\ln\left|\csc\frac{\pi}{6} - \cot\frac{\pi}{6}\right| + \ln\left|\sin\frac{\pi}{6}\right|\right)$$

$$= \left(\ln\lvert\sqrt{2} - 1\rvert + \ln\left|\frac{\sqrt{2}}{2}\right|\right) - \left(\ln\lvert 2 - \sqrt{3}\rvert + \ln\left|\frac{1}{2}\right|\right)$$

$$= \ln\left|\frac{(\sqrt{2}-1)\times\frac{\sqrt{2}}{2}}{(2-\sqrt{3})\times\frac{1}{2}}\right| = \mathbf{\ln\frac{2-\sqrt{2}}{2-\sqrt{3}}}$$

Exercise Evaluate and simplify the following.

- $\int_0^{2\pi} 1\, dx$ [Answer: 2π]
- $\int \left(\sqrt{t} - \frac{2}{t^3}\right) dt$ [Answer: $\frac{2}{3}t\sqrt{t} + \frac{1}{t^2} + C$]
- $\int \sqrt{x}\,(4 - x^2)\, dx$ [Answer: $\frac{8}{3}x^{3/2} - \frac{2}{7}x^{7/2} + C$]

- $\int_0^3 x^5\,dx$ [Answer: $\frac{243}{2}$]
- $\int_0^{2a} x^2\,dx$ [Answer: $\frac{8}{3}a^3$]
- $\int_0^{\sqrt{a}} 4y^2\,dy$ [Answer: $\frac{4}{3}a\sqrt{a}$]
- $\int_{2a}^{4-a} a^2x\,dx$ [Answer: $\frac{1}{2}a^2(16-8a-3a^2)$]
- $\int (8t^3+5t^2-2)\,dt$ [Answer: $2t^4+\frac{5}{3}t^3-2t+C$]
- $e^{\int \frac{4}{x}dx}$ [Answer: Cx^4]
- $\int (\cos x - 2ax)\,dx$ [Answer: $\sin x - ax^2 + C$]
- $\int_1^3 \left(t^2+\frac{1}{t^2}\right) dt$ [Answer: $\frac{28}{3}$]

Substitution

Example Evaluate and simplify $\int_0^2 r\sqrt{1+4r^2}\,dr$.

Solution:

(We can solve this problem by using a simple direct substitution $u = 1+4r^2$)

$u = 1+4r^2 \Longrightarrow du = \frac{du}{dr}\cdot dr = 8r\cdot dr \Longrightarrow r\cdot dr = \frac{1}{8}du;$

$r{=}0, u{=}1{+}4{\times}0^2 = 1; \quad r = 2, u = 1+4\times 2^2 = 17;$

$\int_0^2 r\sqrt{1+4r^2}\,dr = \int_0^2 \sqrt{1+4r^2}\cdot r\,dr = \int_1^{17}\sqrt{u}\cdot\frac{1}{8}\,du = \frac{1}{8}\cdot\frac{u^{3/2}}{3/2}\Big|_1^{17}$

$=\frac{1}{12}\left(17^{3/2}-1^{3/2}\right) = \mathbf{\frac{1}{12}\left(17\sqrt{17}-1\right)}$

Example Evaluate and simplify $\int_0^{\pi/8} \frac{\sec^2(2x)}{1+\tan(2x)}\,dx$.

Solution:

$u = 1+\tan(2x) \Rightarrow du = \frac{du}{dx}\cdot dx = \sec^2(2x)\cdot 2\,dx = 2\sec^2(2x)\;dx$

$\Rightarrow \sec^2(2x)\;dx = \frac{1}{2}du;$

$x = 0, u = 1+\tan(2\cdot 0) = 1+0 = 1;\;\; x = \frac{\pi}{8}, u = 1+\tan\left(2\cdot\frac{\pi}{8}\right) = 1+1 = 2;$

$\int_0^{\pi/8}\frac{\sec^2(2x)}{1+\tan(2x)}\,dx = \int_1^2 \frac{1}{u}\cdot\frac{1}{2}du = \frac{1}{2}\ln|u|\Big|_1^2 = \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 2 - 0 = \mathbf{\frac{1}{2}\ln 2}$

Exercise Evaluate and simplify the following.

- $\int x^3(2x^4+1)^5\,dx$ [Answer: $\frac{1}{48}(2x^4+1)^6+C$]
- $\int [2x-(1-x)^4]\,dx$ [Answer: $x^2+\frac{1}{5}(1-x)^5+C$]
- $\int_0^1 5t(t^2+2)^5\,dt$ [Answer: $\frac{3325}{12}$]
- $\int x\sqrt{1-2x^2}\,dx$ [Answer: $-\frac{1}{6}(1-2x^2)^{3/2}+C$]
- $\int \frac{x}{\sqrt{4x^2-1}}\,dx$ [Answer: $\frac{1}{4}\sqrt{4x^2-1}+C$]
- $\int \frac{5x^2}{(2x^3+1)^5}\,dx$ [Answer: $\frac{-5}{24(2x^3+1)^4}+C$]
- $\int_t^1 \frac{1}{\sqrt{3-x}}\,dx$ [Answer: $2\sqrt{3-t}-2\sqrt{2}$]
- $\int_0^2 r(4+r^2)\,dr$ [Answer: 12]
- $\int_0^1 t\sqrt{t^2+4}\,dt$ [Answer: $\frac{1}{3}\left(5\sqrt{5}-8\right)$]
- $\int_0^3 (4t+1)\sqrt{16t^2+8t+5}\,dt$ [Answer: $\frac{1}{12}\left(173\sqrt{173}-5\sqrt{5}\right)$]
- $\int u^2\sqrt{u^3+2}\,du$ [Answer: $\frac{2}{9}(u^3+2)^{3/2}+C$]
- $\int \left(\frac{1}{x-3}-\frac{2}{x+3}\right)dx$ [Answer: $\ln\frac{|x-3|}{(x+3)^2}+C$]
- $\int \sin(2t)\,dt$ [Answer: $-\frac{1}{2}\cos(2t)+C$]
- $\int \tan(4x)\,dx$ [Answer: $-\frac{1}{4}\ln|\cos(4x)|+C$]
- $\int [5x+6\cos(3x)]\,dx$ [Answer: $\frac{5}{2}x^2+2\sin(3x)+C$]
- $\int_0^{12} \frac{x}{6}\cos\left(\frac{x^2}{2}\right)dx$ [Answer: $\frac{1}{6}\sin 72$]
- $\int_0^3 y^3\cos(y^4)\,dy$ [Answer: $\frac{1}{4}\sin 81$]
- $\int_0^9 x^{3/2}\sin\left(x^{5/2}\right)dx$ [Answer: $\frac{2}{5}(1-\cos 243)$]
- $\int \frac{6\csc^2 x}{3+5\cot x}\,dx$ [Answer: $-\frac{6}{5}\ln|3+5\cot x|+C$]
- $\int \frac{x}{5-x^2}\,dx$ [Answer: $-\frac{1}{2}\ln|5-x^2|+C$]
- $\int \frac{e^{3x}}{1+5e^{3x}}\,dx$ [Answer: $\frac{1}{15}\ln(1+5e^{3x})+C$]
- $\int (\sec x-\sec^3 x\tan x)\,dx$ [Answer: $\ln|\sec x+\tan x|-\frac{1}{3}\sec^3 x+C$]

- $\int e^{-t}\,dt$ [Answer: $-e^{-t}+C$]
- $\int e^{2t}\,dt$ [Answer: $\frac{1}{2}e^{2t}+C$]
- $\int_0^{\pi/2} \mathrm{e}^{\cos(2t)}\sin(2t)\;dt$ [Answer: $\frac{1}{2}\left(e-\frac{1}{e}\right)$]
- $\int_1^4 e^{2u}\,du$ [Answer: $\frac{1}{2}(e^8-e^2)$ or $\frac{e^2}{2}(e^6-1)$]
- $\int_0^1 ye^{-y^2}\,dy$ [Answer: $\frac{1}{2e}(e-1)$]
- $\int \frac{6e^{3x}-5e^x}{e^{x+1}}\,dx$ [Answer: $3e^{2x-1}-\frac{5}{e}x+C$]
- $\int \sin^3 x\;dx$ [Answer: $\frac{1}{3}\cos^3 x-\cos x+C$]
- $\int_0^{\pi/4} \sin\theta\cos^2\theta\;d\theta$ [Answer: $\frac{4-\sqrt{2}}{12}$]
- $\int_0^{\tan^{-1}(1/4)} \sin\theta\cos\theta\;d\theta$ [Answer: $\frac{1}{34}$]
- $\int_0^{\pi/4} \sec^4 x\tan x\;dx$ [Answer: $\frac{3}{4}$]
- $\int \frac{\tan(3y)}{\cos(3y)}\,dy$ [Answer: $\frac{1}{3}\sec(3y)+C$]
- $\int \tan^3(2x)\;dx$ [Answer: $\frac{1}{4}\tan^2(2x)+\frac{1}{2}\ln|\cos(2x)|+C$]
- $\int \frac{x}{(x-2)^3}\,dx$ [Answer: $\frac{1-x}{(x-2)^2}+C$]
- $\int \frac{\sec(e^{-x})}{e^x}\,dx$ [Answer: $-\ln|\sec(e^{-x})+\tan(e^{-x})|+C$]
- $\int x\sec^2(x^2)\;dx$ [Answer: $\frac{1}{2}\tan(x^2)+C$]
- $\int 3\sec^2 t\sqrt{1+\tan t}\;dt$ [Answer: $2(1+\tan t)^{3/2}+C$]
- $\int \frac{\ln x}{x}\,dx$ [Answer: $\frac{1}{2}(\ln x)^2+C$]
- $\int_0^{0.5} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ [Answer: $\frac{1}{72}\pi^2$]

Inverse Trigonometric Forms

$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$	$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

Example The electric current in a certain circuit is given by $i = \int \frac{6t+1}{4t^2+9}dt$, where t is the time.

Integrate and find the resulting function if $i = 0$ for $t = 0$.

Solution:

$i = \int \frac{6t+1}{4t^2+9}dt = \int \frac{6t}{4t^2+9}dt + \int \frac{1}{4t^2+9}dt = 6\int \frac{t}{4t^2+9}dt + \int \frac{1}{4t^2+9}dt$

(For the first integral, let $u = 4t^2 + 9 \Rightarrow du = \frac{du}{dt} \cdot dt = 8t\ dt \Rightarrow t\ dt = \frac{1}{8}du$;

for the second integral, let $v = 2t \Longrightarrow dv = \frac{dv}{dt} \cdot dt = 2\ dt \Rightarrow dt = \frac{1}{2}dv$)

$\Longrightarrow i = 6\int \frac{1}{u} \cdot \frac{1}{8}du + \int \frac{1}{v^2+3^2} \cdot \frac{1}{2}dv = \frac{6}{8}\ln|u| + \frac{1}{2} \cdot \frac{1}{3}\tan^{-1}\frac{v}{3} + C$

$\Longrightarrow i = \frac{3}{4}\ln|4t^2 + 9| + \frac{1}{6}\tan^{-1}\frac{2t}{3} + C = \frac{3}{4}\ln(4t^2 + 9) + \frac{1}{6}\tan^{-1}\frac{2t}{3} + C$

(Use the given condition to find the constant of integration C)

$i = 0$ when $t = 0 \Longrightarrow 0 = \frac{3}{4}\ln(0 + 9) + \frac{1}{6}\tan^{-1}\frac{0}{3} + C$

$\Longrightarrow 0 = \frac{3}{4}\ln 9 + C \Longrightarrow C = -\frac{3}{4}\ln 9$

Hence $i = \frac{3}{4}\ln(4t^2 + 9) + \frac{1}{6}\tan^{-1}\frac{2t}{3} - \frac{3}{4}\ln 9 \Longrightarrow i = \frac{3}{4}[\ln(4t^2 + 9) - \ln 9] + \frac{1}{6}\tan^{-1}\frac{2t}{3}$

$\Longrightarrow \boldsymbol{i = \frac{3}{4}\ln\left(\frac{4t^2+9}{9}\right) + \frac{1}{6}\tan^{-1}\frac{2t}{3}}$

Exercise

- Evaluate and simplify the following.
  - $\int \frac{dx}{\sqrt{16-9x^2}}$ [Answer: $\frac{1}{3}\sin^{-1}\frac{3x}{4} + C$]
  - $\int \frac{1}{4x^2-12x+34}\,dx$ [Answer: $\frac{1}{10}\tan^{-1}\left(\frac{2x-3}{5}\right) + C$]

Other Trigonometric Forms

Example Evaluate and simplify $\int_0^{\pi/6}(\sin\theta - 3\sin\theta\cos\theta + 4\cos^2\theta)\,d\theta$.

Solution:

("Divide" the given integral into three integrals and "conquer" each separately:

use standard formula for the first integral, for the second integral, use either of

the substitutions $u = \sin\theta$ or $u = \cos\theta$; for the third integral, use the power reducing

formula from trigonometry $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$)

First integral: $\int_0^{\pi/6}\sin\theta\ d\theta = -\cos\theta\big|_0^{\pi/6} = \left(-\cos\frac{\pi}{6}\right) - (-\cos 0) = -\frac{\sqrt{3}}{2} + 1;$

Second integral: $u = \sin\theta \Rightarrow du = \frac{du}{d\theta}\cdot d\theta = \cos\theta\cdot d\theta;$

$\theta=0, u=0;\ \theta=\frac{\pi}{6}, \mathrm{u} = \sin\frac{\pi}{6} = \frac{1}{2}$

Hence $\int_0^{\pi/6}(3\sin\theta\cos\theta)\,d\theta = 3\int_0^{\pi/6}\sin\theta\cos\theta\cdot\ d\theta = 3\int_0^{1/2}u\,du$

$= \frac{3}{2}u^2\Big|_0^{1/2} = \frac{3}{2}\left(\frac{1}{2}\right)^2 - \frac{3}{2}(0)^2 = \frac{3}{8} - 0 = \frac{3}{8};$

Third integral: $\int_0^{\pi/6}(4\cos^2\theta)\,d\theta = 4\int_0^{\pi/6}\cos^2\theta\ d\theta = 4\int_0^{\pi/6}\left[\frac{1+\cos(2\theta)}{2}\right]\,d\theta$

$= 2\int_0^{\pi/6}[1+\cos(2\theta)]\,d\theta = 2\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi/6}$

$=2\left\{\left[\frac{\pi}{6} + \frac{1}{2}\sin\frac{2\pi}{6}\right] - \left[0 + \frac{1}{2}\sin 0\right]\right\} = 2\left[\frac{\pi}{6} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right] = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

Hence $\int_0^{\pi/6}(\sin\theta - 3\sin\theta\cos\theta + 4\cos^2\theta)\,d\theta = \left(-\frac{\sqrt{3}}{2} + 1\right) - \frac{3}{8} + \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) = \boldsymbol{\frac{5}{8} + \frac{\pi}{3}}$

Example Evaluate and simplify $\int\ \sin^2\theta\cos^2\theta\ d\theta$.

Solution:

(Use power reducing formulas $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ and $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$)

$\int\ \sin^2\theta\cos^2\theta\ d\theta = \int\ \left[\frac{1-\cos(2\theta)}{2}\right]\left[\frac{1+\cos(2\theta)}{2}\right]\ d\theta = \frac{1}{4}\int\ [1-\cos^2(2\theta)]\ d\theta$

(Use power reducing formula $\cos^2 x = \frac{1+\cos(2x)}{2}$ one more time with $x = 2\theta$)

$= \frac{1}{4}\int\ \left[1 - \frac{1+\cos(4\theta)}{2}\right]\ d\theta = \frac{1}{8}\int\ [1-\cos(4\theta)]\ d\theta = \frac{1}{8}\int\ 1\ d\theta - \frac{1}{8}\int\ \cos(4\theta)\ d\theta$

(Simple substitution $u = 4\theta \Rightarrow du = \frac{du}{d\theta}\cdot d\theta = 4\,d\theta \Longrightarrow d\theta = \frac{1}{4}\ du$)

$= \frac{1}{8}\theta - \frac{1}{8}\int\ \cos u\cdot\frac{1}{4}\ du = \frac{1}{8}\theta - \frac{1}{32}\sin u + C = \boldsymbol{\frac{1}{8}\theta - \frac{1}{32}\sin(4\theta) + C}$

Example Evaluate and simplify $\int \tan^5 x \; dx$.

Solution:

(Use the trigonometric identity $\tan^2 x = \sec^2 x - 1$)

Method 1: $\int \tan^5 x \; dx = \int \tan^4 x \cdot \tan x \; dx = \int (\tan^2 x)^2 \cdot \tan x \; dx$

$= \int (\sec^2 x - 1)^2 \cdot \tan x \; dx = \int (\sec^4 x - 2\sec^2 x + 1) \cdot \tan x \; dx$

$= \int (\sec^4 x \tan x - 2\sec^2 x \tan x + \tan x) \, dx$

$= \int (\sec^4 x \tan x) \, dx - 2\int (\sec^2 x \tan x) \, dx + \int \tan x \; dx$

(For the first two integrals, let $u = \sec x \Longrightarrow du = \frac{du}{dx} \cdot dx = \sec x \tan x \; dx$; for the third integral, apply basic formula)

$= \int \sec^3 x \, (\sec x \tan x) \, dx - 2\int \sec x \, (\sec x \tan x) \, dx + \int \tan x \; dx$

$= \int u^3 \, du - 2\int u \, du + \int \tan x \; dx = \frac{1}{4}u^4 - 2 \cdot \frac{1}{2}u^2 - \ln|\cos x| + C$

$= \boldsymbol{\frac{1}{4}\sec^4 x - \sec^2 x - \ln|\cos x| + C_1}$

Method 2: $\int \tan^5 x \; dx = \int \tan^3 x \cdot \tan^2 x \; dx = \int \tan^3 x \cdot (\sec^2 x - 1) \, dx$

$= \int (\tan^3 x \sec^2 x - \tan^3 x) \, dx = \int (\tan^3 x \sec^2 x) \, dx - \int (\tan^3 x) \, dx$

$= \int (\tan^3 x \sec^2 x) \, dx - \int (\tan^2 x) \tan x \; dx$

$= \int (\tan^3 x \sec^2 x) \, dx - \int (\sec^2 x - 1) \tan x \; dx$

$= \int (\tan^3 x \sec^2 x) \, dx - \int (\tan x \sec^2 x - \tan x) \, dx$

$= \int (\tan^3 x \sec^2 x) \, dx - \int (\tan x \sec^2 x) \, dx + \int \tan x \; dx$

(For the first two integrals, let $u = \tan x \Longrightarrow du = \frac{du}{dx} \cdot dx = \sec^2 x \; dx$; for the third integral, use basic formula)

$= \int u^3 \, du - \int u \, du + \int \tan x \; dx = \frac{1}{4}u^4 - \frac{1}{2}u^2 - \ln|\cos x| + C$

$= \boldsymbol{\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C_2}$

Note: the answers obtained by using method 1 and method 2 do not look alike; however, by using the trigonometric identity $\sec^2 x = 1 + \tan^2 x$, one can easily show that they are equivalent with $C_2 = C_1 - \frac{3}{4}$.

Exercise Evaluate and simplify the following.

- $\int \cos^5 2x \; dx$ [Answer: $\frac{1}{2}\sin 2x - \frac{1}{3}\sin^3 2x + \frac{1}{10}\sin^5 2x + C$]
- $\int \sin^2\theta \; d\theta$ [Answer: $\frac{\theta}{2} - \frac{\sin(2\theta)}{4} + C$]
- $\int_0^{\pi/3} \sin^2 3\theta \; d\theta$ [Answer: $\frac{\pi}{6}$]
- $\int \cos^2\theta \; d\theta$ [Answer: $\frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C$]
- $\int \cos^2(4\theta) \; d\theta$ [Answer: $\frac{\theta}{2} + \frac{\sin(8\theta)}{16} + C$]
- $\int \cos^4\theta \; d\theta$ [Answer: $\frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta) + C$]
- $\int \sin^2 t \cos^4 t \; dt$ [Answer: $\frac{t}{16} - \frac{\sin(4t)}{64} + \frac{\sin^3(2t)}{48} + C$]
- $\int_0^{\pi/4} \frac{\tan^3\theta}{\sec^3\theta} \; d\theta$ [Answer: $\frac{8-5\sqrt{2}}{12} \approx 0.077$]

Integration by Parts: $\int u\, dv = uv - \int v\, du$

Example Use integration by parts to show the following.

$$\int \sec^3\theta \; d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C$$

Solution:

(Use integration by parts on $\int \sec^3\theta \; d\theta = \int \sec\theta \sec^2\theta \; d\theta$ with

$u = \sec\theta$ and $dv = \sec^2\theta \; d\theta$, then $du = \left(\frac{d}{d\theta}\sec\theta\right) d\theta = \sec\theta\tan\theta\, d\theta$ and

$v = \int \sec^2\theta \; d\theta = \tan\theta$)

$$\int \sec^3\theta \; d\theta = \int \sec\theta\sec^2\theta \; d\theta = \sec\theta\tan\theta - \int \tan\theta \cdot \sec\theta\tan\theta \; d\theta$$

$$\Rightarrow \int \sec^3\theta \; d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta \; d\theta$$

(Rewrite the right-hand-side using the trigonometric identity $\tan^2\theta = \sec^2\theta - 1$)

$$\Rightarrow \int \sec^3\theta \; d\theta = \sec\theta\tan\theta - \int \sec\theta\,(\sec^2\theta - 1)\, d\theta$$

$$\Rightarrow \int \sec^3\theta \; d\theta = \sec\theta\tan\theta - \int (\sec^3\theta - \sec\theta)\, d\theta$$

$$\Rightarrow \int \sec^3\theta \; d\theta = \sec\theta\tan\theta - \int \sec^3\theta \; d\theta + \int \sec\theta \; d\theta$$

(Collect like terms involving $\int \sec^3\theta \; d\theta$ and solve for the integral)

$$\Rightarrow 2\int \sec^3\theta \; d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C_1$$

$$\Rightarrow \int \sec^3\theta \; d\theta = \boldsymbol{\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C} \quad \text{(where } C = \tfrac{C_1}{2}\text{)}$$

Example Evaluate and simplify $\int x^2 e^{3x}\,dx$.

Solution:

Method 1: (Use integration by parts: $\begin{cases} u = x^2 & \Longrightarrow & du = 2x\,dx \\ dv = e^{3x}dx & \Longrightarrow & v = \int e^{3x}dx = \frac{1}{3}e^{3x}(+C) \end{cases}$)

$$\int x^2 e^{3x}\,dx = (x^2)\left(\tfrac{1}{3}e^{3x}\right) - \int \left(\tfrac{1}{3}e^{3x}\right)\cdot 2x\,dx = \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{3}\int xe^{3x}\,dx$$

(Use integration by parts a second time: $\begin{cases} u = x & \Longrightarrow & du = dx \\ dv = e^{3x}dx & \Longrightarrow & v = \int e^{3x}dx = \frac{1}{3}e^{3x}(+C) \end{cases}$)

$$= \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{3}\left[(x)\left(\tfrac{1}{3}e^{3x}\right) - \int\left(\tfrac{1}{3}e^{3x}\right)\cdot 1\,dx\right] = \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{9}xe^{3x} + \tfrac{2}{9}\int e^{3x}\,dx$$

$$= \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{9}xe^{3x} + \tfrac{2}{9}\left[\tfrac{1}{3}e^{3x} + C_1\right] = \boldsymbol{\tfrac{1}{3}x^2e^{3x} - \tfrac{2}{9}xe^{3x} + \tfrac{2}{27}e^{3x} + C}$$

(or $\boldsymbol{\left(\tfrac{1}{3}x^2 - \tfrac{2}{9}x + \tfrac{2}{27}\right)e^{3x} + C}$)

Method 2: (Use integration by parts in tabular form)

$$\int x^2e^{3x}\,dx$$

$$= +(x^2)\left(\tfrac{1}{3}e^{3x}\right) - (2x)\left(\tfrac{1}{9}e^{3x}\right) + (2)\left(\tfrac{1}{27}e^{3x}\right) - C_1$$

$$= \boldsymbol{\tfrac{1}{3}x^2e^{3x} - \tfrac{2}{9}xe^{3x} + \tfrac{2}{27}e^{3x} + C}$$

(or $\boldsymbol{\left(\tfrac{1}{3}x^2 - \tfrac{2}{9}x + \tfrac{2}{27}\right)e^{3x} + C}$)

Differentiate (D)	Integrate (I)
x^2	e^{3x}
$2x$	$\frac{1}{3}e^{3x}$
2	$\frac{1}{9}e^{3x}$
0	$\frac{1}{27}e^{3x}$

Example Evaluate and simplify $\int e^x \sin x\,dx$.

Solution:

(Perform integration by parts twice; we can either differentiate exponential function and integrate trigonometric function, or vice versa – as long as we do the same twice)

$$\int e^x \sin x\,dx$$

$$= (e^x)(-\cos x) - (e^x)(-\sin x) + \int (e^x)(-\sin x)\,dx$$

$$\Rightarrow \int e^x \sin x\,dx = -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx$$

D	I
e^x	$\sin x$
e^x	$-\cos x$
e^x	$-\sin x$

(Solve the equation for the integral $\int e^x \sin x\,dx$ by collecting like terms)

$$\Rightarrow 2\int e^x \sin x\,dx = -e^x\cos x + e^x\sin x + C_1$$

$\Rightarrow \int e^x \sin x \; dx = -\frac{1}{2}\boldsymbol{e^x \cos x} + \frac{1}{2}\boldsymbol{e^x \sin x + C}$ where $C = \frac{1}{2}C_1$

or $\int e^x \sin x \; dx = \frac{1}{2}\boldsymbol{e^x(\sin x - \cos x) + C}$

Exercise Evaluate and simplify the following.

- $\int x^8 \sin(x^3) \; dx$ [Answer: $\frac{1}{3}(2 - x^6)\cos(x^3) + \frac{2}{3}x^3 \sin(x^3) + C$]
- $\int xe^x \, dx$ [Hint: Use integration by parts] [Answer: $(x - 1)e^x + C$]
- $\int re^{r/2} \, dr$ [Answer: $(2r - 4)e^{r/2} + C$]
- $\int z^2 e^z \, dz$ [Answer: $(z^2 - 2z + 2)e^z + C$]
- $\int \ln u \; du$ [Answer: $u(\ln u - 1) + C$]
- $\int x^2 \ln x \; dx$ [Answer: $\frac{x^3}{9}(3\ln x - 1) + C$]
- $\int_0^{\pi/2} t \sin^2 t \; dt$ [Answer: $\frac{1}{16}(\pi^2 + 4)$]
- $\int \sin^{-1} x \; dx$ [Answer: $x \sin^{-1} x + \sqrt{1 - x^2} + C$]
- $\int x\sqrt{1 - x} \; dx$ [Answer: $-\frac{2}{15}(1 - x)^{3/2}(2 + 3x) + C$]

Trigonometric Substitution

- For $\sqrt{a^2 - u^2}$, use $u = a \sin\theta$
- For $\sqrt{a^2 + u^2}$, use $u = a\tan\theta$
- For $\sqrt{u^2 - a^2}$, use $u = a \sec\theta$

Example Evaluate and simplify $\int \frac{2}{x\sqrt{x^2-9}} \; dx$.

Solution:

(Due to the radical $\sqrt{x^2 - 9} = \sqrt{x^2 - 3^2}$, we can try using the trigonometric substitution $x = 3\sec\theta$ and rewrite the integral using the new variable θ)

$x = 3\sec\theta \Longrightarrow dx = \frac{dx}{d\theta}d\theta = 3\sec\theta\tan\theta \; d\theta$

$\Longrightarrow \int \frac{2}{x\sqrt{x^2-9}} \; dx = \int \frac{2}{(\cancel{3\sec\theta})\sqrt{(3\sec\theta)^2-9}} \cdot \cancel{3\sec\theta}\tan\theta \; d\theta = \int \frac{2\tan\theta}{\sqrt{9\sec^2\theta-9}} \; d\theta$

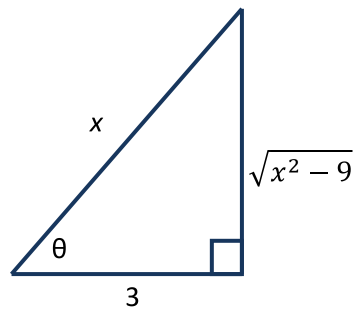
(Simplify using the trigonometric identity $\sec^2\theta = 1 + \tan^2\theta$ and then integrate)

$$= \int \frac{2\tan\theta}{\sqrt{9(\sec^2\theta - 1)}}\,d\theta = \int \frac{2\tan\theta}{\sqrt{9\tan^2\theta}}\,d\theta = \int \frac{2\cancel{\tan\theta}}{3\cancel{\tan\theta}}\,d\theta \quad \text{(assuming } \tan\theta > 0\text{)}$$

$$= \frac{2}{3}(\theta + C_1) = \frac{2}{3}\theta + C$$

(Rewrite the answer using the original variable x)

$$= \mathbf{\frac{2}{3}\sec^{-1}\frac{x}{3} + C} \quad \left(\text{or} \quad \mathbf{\frac{2}{3}\cos^{-1}\frac{3}{x} + C}\right)$$



Exercise Evaluate and simplify the following.

- $\int \frac{dx}{\sqrt{16+x^2}}$ [Answer: $\ln\left(\sqrt{16 + x^2} + x\right) + C$]
- $\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$ [Answer: $-\frac{\sqrt{4-x^2}}{4x} + C$]

Partial Fractions

Example Evaluate $\int_5^{10} \frac{1}{4x^2-12x-7}\,dx$.

Solution:

(Identify the form of the partial fraction decomposition: check that the discriminant of the denominator is a perfect square and hence can be factored using the *ac*-method)

Discriminant $D = b^2 - 4ac = (-12)^2 - 4(4)(-7) = 144 + 112 = 256 = 16^2$

$\Longrightarrow 4x^2 - 12x - 7 = 4x^2 - 14x + 2x - 7$

$= 2x(2x - 7) + (2x - 7) = (2x - 7)(2x + 1)$

$$\frac{1}{4x^2-12x-7} = \frac{1}{(2x-7)(2x+1)} = \frac{A}{2x-7} + \frac{B}{2x+1}$$

(Determine the constants, by using the method of undetermined coefficients, for instance)

$1 = A(2x + 1) + B(2x - 7)$

$\Rightarrow 1 = 2Ax + A + 2Bx - 7B$

$\Rightarrow 1 = (2A + 2B)x + (A - 7B)$

$$\Rightarrow \left\{\begin{matrix} 2A + 2B = 0 & \Rightarrow & 2(A + B) = 0 & \Rightarrow & A + B = 0 \\ A - 7B = 1 & \Rightarrow & A = 7B + 1 & & \end{matrix}\right\}$$

$$\Longrightarrow (7B + 1) + B = 0 \Rightarrow 8B = -1 \Rightarrow B = -\frac{1}{8}; A = 7\left(-\frac{1}{8}\right) + 1 = \frac{1}{8}$$

$$\frac{1}{4x^2 - 12x - 7} = \frac{(1/8)}{2x-7} + \frac{(-1/8)}{2x+1}$$

(To integrate the partial fraction decomposition, we can use simple substitutions

$u = 2x - 7, du = 2dx$ and $u = 2x + 1, du = 2dx$)

$$\int_5^{10} \frac{1}{4x^2-12x-7}\, dx = \int_5^{10} \frac{(1/8)}{2x-7}\, dx + \int_5^{10} \frac{(-1/8)}{2x+1}\, dx$$

$$= \frac{1}{8(2)} \int_5^{10} \frac{2}{2x-7}\, dx - \frac{1}{8(2)} \int_5^{10} \frac{2}{2x+1}\, dx$$

$$= \frac{1}{16}[\ln|2x-7|]_5^{10} - \frac{1}{16}[\ln|2x+1|]_5^{10}$$

$$= \frac{1}{16}(\ln 13 - \ln 3) - \frac{1}{16}(\ln 21 - \ln 11) = \frac{1}{16}\left(\ln\frac{13}{3} - \ln\frac{21}{11}\right)$$

$$= \frac{1}{16}\ln\frac{13/3}{21/11} = \frac{1}{16}\ln\frac{13\cdot 11}{3\cdot 21} = \mathbf{\frac{1}{16}\ln\frac{143}{63}} (\approx 0.05123)$$

Exercise Evaluate and simplify the following.

- $\int \frac{1}{s(s+2)}\, ds$ [Answer: $\frac{1}{2}\ln|s| - \frac{1}{2}\ln|s+2| + C$ or $\frac{1}{2}\ln\left|\frac{s}{s+2}\right| + C$]
- $\int \frac{7-x}{x^2+x-2}\, dx$ [Answer: $2\ln|x-1| - 3\ln|x+2| + C$ or $\ln\left|\frac{(x-1)^2}{(x+2)^3}\right| + C$]
- $\int_0^2 \frac{23x+11}{4x^2+9x+2}\, dx$ [Answer: $5\ln 2 + \frac{3}{2}\ln 3$]
- $\int \frac{4y-13}{2y^2+y-6}\, dy$ [Answer: $3\ln|y+2| - \ln|2y-3| + C$ or $\ln\left|\frac{(y+2)^3}{(2y-3)}\right| + C$]
- $\int \frac{6t^3+5t^2-7}{3t^2-2t-1}\, dt$ [Answer: $t^2 + 3t + \frac{5}{3}\ln|3t+1| + \ln|t-1| + C$]
- $\int \frac{6x^2-14x-11}{(x+1)(x-2)(2x+1)}\, dx$

  [Answer: $3\ln|x+1| - \ln|x-2| + \ln|2x+1| + C$ or $\ln\left|\frac{(x+1)^3(2x+1)}{(x-2)}\right| + C$]
- $\int \frac{2x^4-x^3-9x^2+x-12}{x^3-x^2-6x}\, dx$

  [Answer: $x^2 + x + 2\ln|x| - \ln|x+2| + 3\ln|x-3| + C$ or $x^2 + x + \ln\left|\frac{x^2(x-3)^3}{x+2}\right| + C$]

Exercise Evaluate and simplify the following.

- $\int \frac{dx}{x(x+3)^2}$ [Answer: $\frac{1}{9}\ln|x| - \frac{1}{9}\ln|x+3| + \frac{1}{3(x+3)} + C$ or $\frac{1}{9}\ln\left|\frac{x}{x+3}\right| + \frac{1}{3(x+3)} + C$]
- $\int \frac{7x^2-29x+24}{(2x-1)(x-2)^2}\, dx$ [Answer: $\frac{5}{2}\ln|2x-1| + \ln|x-2| + \frac{2}{x-2} + C$]
- $\int \frac{3x^3+15x^2+21x+15}{(x-1)(x+2)^3}\, dx$

  [Answer: $2\ln|x-1| + \ln|x+2| + \frac{3}{2(x+2)^2} + C$ or $\ln|(x-1)^2(x+2)| + \frac{3}{2(x+2)^2} + C$]

Example Evaluate $\int \frac{14x^2+27x+99}{(5x-1)(x^2+6x+25)}\,dx$.

Solution:

(Identify the form of the partial fraction decomposition)

$$\frac{14x^2+27x+99}{(5x-1)(x^2+6x+25)} = \frac{A}{5x-1} + \frac{Bx+C}{x^2+6x+25}$$

(Determine the constants, by using the method of undetermined coefficients, for instance)

$$14x^2+27x+99 = A(x^2+6x+25)+(Bx+C)(5x-1)$$

$$\Rightarrow 14x^2+27x+99 = Ax^2+6Ax+25A+5Bx^2+5Cx-Bx-C$$

$$\Rightarrow 14x^2+27x+99 = (A+5B)x^2+(6A-B+5C)x+(25A-C)$$

$$\Rightarrow \left\{\begin{matrix} A+5B=14 \\ 6A-B+5C=27 \\ 25A-C=99 \quad \Longrightarrow \quad C=25A-99 \end{matrix}\right\}$$

$$\Rightarrow \left\{\begin{matrix} A+5B=14 \\ 6A-B+5(25A-99)=27 \quad \Rightarrow \quad 6A-B+125A-495=27 \end{matrix}\right\}$$

$$\Rightarrow \left\{\begin{matrix} A+5B=14 \\ 131A-B=522 \quad \Rightarrow \quad B=131A-522 \end{matrix}\right\}$$

$$\Longrightarrow A+5(131A-522)=14 \Rightarrow A+655A-2610=14$$

$$\Rightarrow 656A=2624 \Rightarrow A=\frac{2624}{656}=4;$$

$$B=131(4)-522=524-522=2;$$

$$C=25(4)-99=100-99=1;$$

$$\frac{14x^2+27x+99}{(5x-1)(x^2+6x+25)} = \frac{4}{5x-1} + \frac{2x+1}{x^2+6x+25}$$

(To integrate the partial fraction decomposition, for the first term, we can use a simple substitution $u=5x-1, du=5dx$; for the second term, the substitution $u=x^2+6x+25$ leads to $du=(2x+6)dx$, hence we may try rewriting the numerator as $(2x+6)-5$ and split the second integral into two)

$$\int \frac{14x^2+27x+99}{(5x-1)(x^2+6x+25)}\,dx = \int \left[\frac{4}{5x-1} + \frac{2x+1}{x^2+6x+25}\right] dx$$

$$= \int \left[\frac{4}{5x-1} + \frac{(2x+6)-5}{x^2+6x+25}\right] dx = \int \left[\frac{4}{5x-1} + \frac{2x+6}{x^2+6x+25} - \frac{5}{x^2+6x+25}\right] dx$$

(To integrate the third term, we rewrite the denominator by completing the square)

$$x^2+6x+25 = \left[x^2+6x+\left(\frac{6}{2}\right)^2\right]+25-\left(\frac{6}{2}\right)^2$$

$$= [x^2+6x+9]+25-9 = (x+3)^2+16$$

$$\int \frac{4}{5x-1}\,dx + \int \frac{2x+6}{x^2+6x+25}\,dx - \int \frac{5}{(x+3)^2+16}\,dx$$

$$= \frac{4}{5}\int \frac{5}{5x-1}\,dx + \int \frac{2x+6}{x^2+6x+25}\,dx - \int \frac{5}{(x+3)^2+4^2}\,dx$$

$$= \boldsymbol{\frac{4}{5}\ln|5x-1| + \ln(x^2+6x+25) - \frac{5}{4}\tan^{-1}\frac{x+3}{4} + C}$$

(Note that since $x^2 + 6x + 25 = (x+3)^2 + 16 > 0$, absolute value is not needed in the result of the second integral.)

Exercise Evaluate and simplify the following.

- $\int \frac{1}{(s+2)(s^2+4)}\,ds$

  [Answer: $\frac{1}{8}\ln|s+2| - \frac{1}{16}\ln(s^2+4) + \frac{1}{8}\tan^{-1}\frac{s}{2} + C$ or $\frac{1}{16}\ln\left|\frac{(s+2)^2}{s^2+4}\right| + \frac{1}{8}\tan^{-1}\frac{s}{2} + C$]

- $\int \frac{x^3+3x^2+2x+4}{x^2(x^2+2x+2)}\,dx$ [Answer: $\ln(x^2+2x+2) - \ln|x| - \frac{2}{x} + \tan^{-1}(x+1) + C$ or $\ln\left|\frac{x^2+2x+2}{x}\right| - \frac{2}{x} + \tan^{-1}(x+1) + C$]

- $\int \frac{x^3+5x^2+x+2}{x^4+x^2}\,dx$ [Answer: $\ln|x| - \frac{2}{x} + 3\tan^{-1}x + C$]