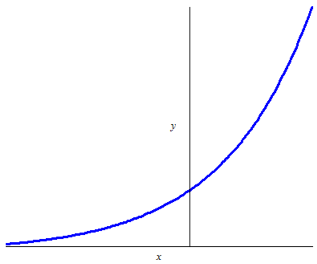
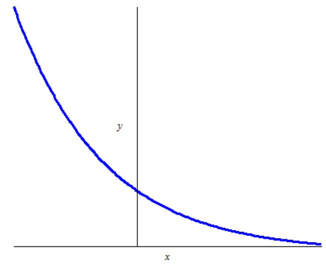
Exponential and Logarithmic Functions

<u>Exponential function with base *b*</u>: $y = b^x$



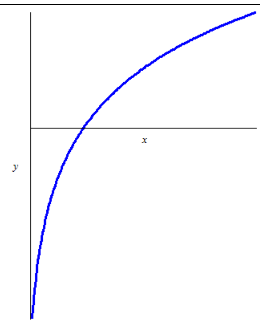
$(b > 1)$



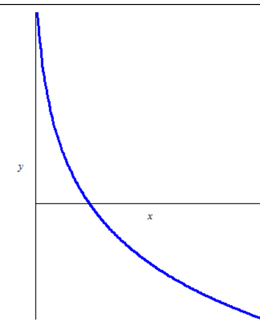
$(0 < b < 1)$

(Exponential function with base *e*: $y = e^x$)

<u>Logarithmic function with base *b*</u>: $y = \log_b x$



$(b > 1)$



$(0 < b < 1)$

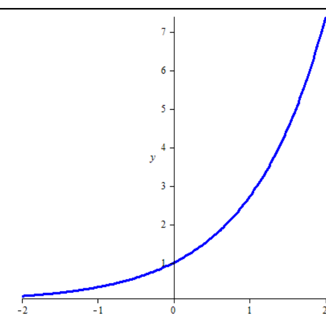
Common Logarithmic Function (Logarithmic function with base 10): $y = \log_{10} x = \log x$

Natural Logarithmic Function (Logarithmic function with base *e*): $y = \log_e x = \ln x$

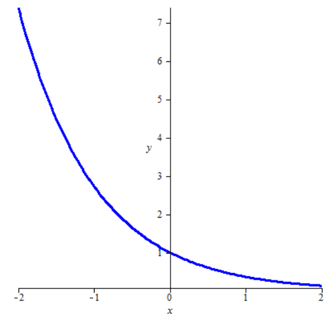
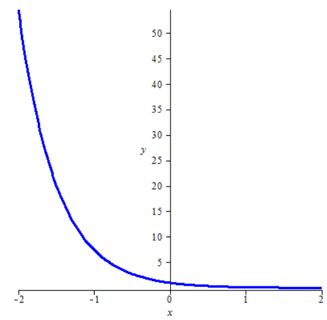
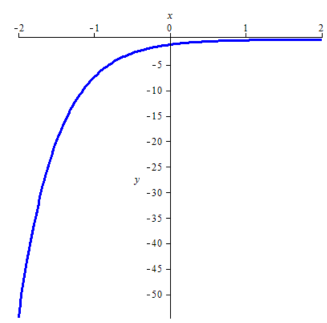
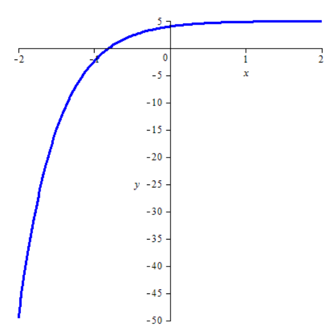
<u>Example</u> Sketch the graph of the function $y = 5 - e^{-2x}$.

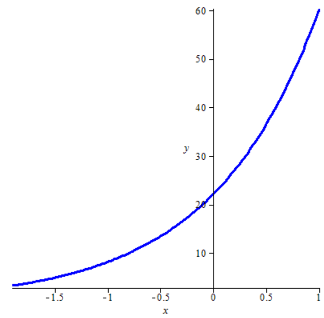
Solution:

$y = e^x$ (Standard Function)



Exponential and Logarithmic Functions

$y = e^{-x}$ (Replace *x* by –*x*)	(Horizontal flip about the *y*-axis) 
$y = e^{-2x}$ (Multiply *x* by 2)	(Expand vertically by factor of 2) 
$y = -e^{-2x}$ (Multiply function by –1)	(Vertical flip about the *x*-axis) 
$y = 5 - e^{-2x}$ (Add 5 to the function)	(Up-shift the graph 5 units) 

Exercise Sketch the graph of the function $y = 3e^{x+2}$. [Answer: 

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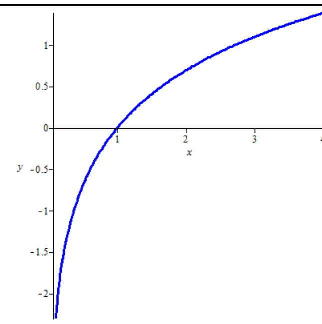
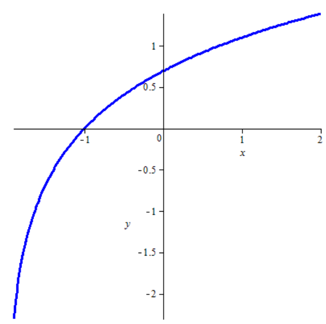
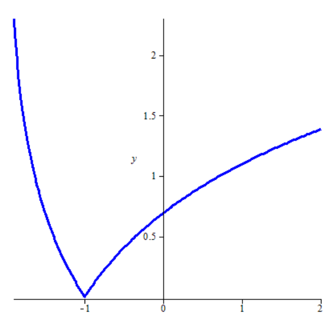
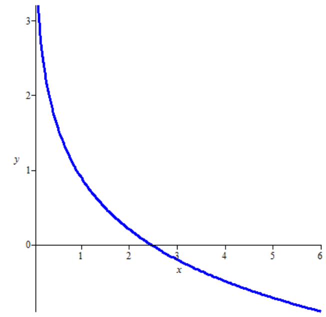
Example Sketch the graph of the function $y = |\ln(x+2)|$.

Solution:

$y = \ln x$ (Standard Function)	
$y = \ln(x+2)$ (Replace x by $x+2$)	(Left-shift the graph 2 units) 
$y = \|\ln(x+2)\|$ (Put absolute value on the function)	(Vertically flip any negative portion of the graph below the x-axis) 

Exponential and Logarithmic Functions

Exercise Sketch the graph of the function $y = 2 - \ln(3x)$. [Answer: 

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Properties of logarithms:

- $\log_b(xy) = \log_b x + \log_b y$
- $\log_b \frac{x}{y} = \log_b x - \log_b y$
- $\log_b(x^n) = n \log_b x$
- $\log_b 1 = 0$
- $\log_b b = 1$
- Changing bases of logarithms: $\log_b x = \frac{\log_a x}{\log_a b}$

  In particular, $\ln x = \frac{\log x}{\log e}$ and $\log x = \frac{\ln x}{\ln 10}$.

Example Write $\log_2(3n^4)$ as sum or difference of multiples of logarithms.

Solution:

(Use the properties of logarithms)

$\log_2(3n^4) = \log_2(3) + \log_2(n^4) = \log_2(3) + 4\log_2(n) = \mathbf{\log_2 3 + 4\log_2 n}$

Exercise

- Write $\log_x\left(\sqrt[3]{\frac{x^2 y}{z^7}}\right)$ as sum or difference of multiples of logarithms.

  [Answer: $\frac{2}{3} + \frac{1}{3}\log_x y - \frac{7}{3}\log_x z$]

Exponential and Logarithmic Functions

Exponential function and logarithmic function are inverse functions:

$$b^{\log_b x} = x; \quad \log_b(b^x) = x$$

Example Solve the equation $1000 - 850e^{-\frac{t}{100}} = 500$.

Solution:

(Since the variable t appears in the exponent, we first rewrite the equation to isolate $e^{-\frac{t}{100}}$)

$1000 - 850e^{-\frac{t}{100}} = 500 \Rightarrow 1000 - 500 = 850e^{-\frac{t}{100}}$

$\Rightarrow 850e^{-\frac{t}{100}} = 50 \Rightarrow e^{-\frac{t}{100}} = \frac{500}{850} \Rightarrow e^{-\frac{t}{100}} = \frac{10}{17}$

(Apply the natural logarithmic function, the inverse function of the exponential function, to both sides of the equation; the inverse functions cancel each other)

$\ln\left(e^{-\frac{t}{100}}\right) = \ln\frac{10}{17} \Longrightarrow -\frac{t}{100} = \ln\frac{10}{17} \Rightarrow \boldsymbol{t = -100\ln\frac{10}{17} \approx 53.063}$

Example Solve the equation $\frac{y-2x-3}{y-2x+3} = Ce^{6x}$ for y. [Answer: $y = 2x + \frac{3(1+Ce^{6x})}{1-Ce^{6x}}$]

Solution:

(Clear the denominator) $y - 2x - 3 = Ce^{6x}(y - 2x + 3)$

(Expand the right-hand-side) $y - 2x - 3 = Ce^{6x}y - 2Cxe^{6x} + 3Ce^{6x}$

(Bring all the terms involving y to one side of the equation and the rest to the other side)

$$y - Ce^{6x}y = -2Cxe^{6x} + 3Ce^{6x} + 2x + 3$$

(Factor out common factor y from the left hand side)

$$y(1 - Ce^{6x}) = 2x - 2Cxe^{6x} + 3 + 3Ce^{6x}$$

(Solve for y and simplify)

$$y = \frac{2x - 2Cxe^{6x} + 3 + 3Ce^{6x}}{1 - Ce^{6x}} = \frac{2x(1 - Ce^{6x}) + 3(1 + Ce^{6x})}{1 - Ce^{6x}} = \boldsymbol{2x + \frac{3(1 + Ce^{6x})}{1 - Ce^{6x}}}$$

Exercise Solve the equation

- $e^{3x} = 15$ [Answer: $x = \frac{1}{3}\ln(15) \approx 0.903$]
- $3^x = 20$ [Answer: $x = \frac{\ln 20}{\ln 3} \approx 2.7268$]

Example Solve the equation $\ln(x+1) = 3$.

Solution:

(Use both sides of the equation as arguments of the base-*e* exponential function and simplify using the property of inverse functions)

$$e^{\ln(x+1)} = e^3 \Longrightarrow x + 1 = e^3 \Longrightarrow \boldsymbol{x = e^3 - 1 \approx 19.086}$$

Exercise Solve the equation.

- $\log_3(2x-1) - \log_3(x-4) = 2$ [Answer: $x = 5$]