Differentiation

Applications of Derivatives: Tangents

$\Delta x = x_2 - x_1$; $\Delta y = y_2 - y_1 = f(x_1 + \Delta x) - f(x_1)$

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

$$m_{\tan} = \lim_{\Delta x \to 0} m_{PQ} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

Definition of Derivative: Delta Process

$\Delta y = (y + \Delta y) - y = f(x + \Delta x) - f(x)$

$$f'(x)\left(\text{or } \frac{dy}{dx}\right) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Example Find the derivative of $y = 2x^3 + \frac{4}{x}$ by using the definition.

Solution:

$$f(x) = 2x^3 + \frac{4}{x}$$

$$\Longrightarrow f(x + \Delta x) = 2(x + \Delta x)^3 + \frac{4}{x + \Delta x} = 2[x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3] + \frac{4}{x + \Delta x}$$

$$\Longrightarrow f(x + \Delta x) - f(x) = 2[x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3] + \frac{4}{x + \Delta x} - 2x^3 - \frac{4}{x}$$

$$= \cancel{2x^3} + 6x^2(\Delta x) + 6x(\Delta x)^2 + 2(\Delta x)^3 - \cancel{2x^3} + \frac{4}{x + \Delta x} - \frac{4}{x}$$

$$= 6x^2(\Delta x) + 6x(\Delta x)^2 + 2(\Delta x)^3 + \frac{4x - 4(x + \Delta x)}{x(x + \Delta x)}$$

$$= 6x^2(\Delta x) + 6x(\Delta x)^2 + 2(\Delta x)^3 + \frac{\cancel{4x} - \cancel{4x} - 4(\Delta x)}{x(x + \Delta x)}$$

$$= 6x^2(\Delta x) + 6x(\Delta x)^2 + 2(\Delta x)^3 - \frac{4(\Delta x)}{x(x + \Delta x)}$$

$$\Longrightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \left[\frac{6x^2(\Delta x)}{(\Delta x)} + \frac{6x(\Delta x)^2}{(\Delta x)} + \frac{2(\Delta x)^3}{(\Delta x)} - \frac{4(\Delta x)}{x(x + \Delta x)(\Delta x)}\right]$$

$$= \lim_{\Delta x \to 0} \left[6x^2 + 6x(\Delta x) + 2(\Delta x)^2 - \frac{4}{x(x + \Delta x)}\right]$$

$$= \lim_{\Delta x \to 0} (6x^2) + \lim_{\Delta x \to 0} [6x(\Delta x)] + \lim_{\Delta x \to 0} [2(\Delta x)^2] - \lim_{\Delta x \to 0} \frac{4}{x(x + \Delta x)}$$

$$= 6x^2 + [6x(0)] + [2(0)^2] - \frac{4}{x(x + 0)} = \mathbf{6x^2 - \frac{4}{x^2}}$$

Exercise Find the derivative dy/dx by using the definition.

- $y = 3x^2 + 2x$ [Answer: $\frac{dy}{dx} = 6x + 2$]
- $y = 5x - 2x^2$ [Answer: $\frac{dy}{dx} = 5 - 4x$]
- $y = \frac{3}{x+2}$ [Answer: $\frac{dy}{dx} = -\frac{3}{(x+2)^2}$]
- $y = \sqrt{x}$ [Answer: $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$]

Basic Differentiation Formulas

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(cu) = c\frac{d}{dx}(u)$
- $\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Derivative of Polynomials

Basic Formula	Basic Formula with Chain Rule
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

Exercise Evaluate and simplify the following derivatives.

- $\frac{d}{dx}(ax^2 - x^2 - 4ax + 4x + 5a)$ [Answer: $2ax - 2x - 4a + 4$]
- $\frac{d}{dx}(11x^2 + 14ax + 4a^2 - 48x - 36a + 5)$ [Answer: $22x + 14a - 48$]
- $\frac{d}{dx}\left(\frac{2+3x}{5}\right)$ [Answer: $\frac{3}{5}$]
- $\frac{d}{dx}(4x^6 - 2x^4 + \pi^3)$ [Answer: $24x^5 - 8x^3$]

Example $\frac{d}{dx}\sum_{n=0}^{N}(-1)^n x^n$

Solution:

(Derivative of a sum is the sum of derivatives)

$$\frac{d}{dx}\sum_{n=0}^{N}(-1)^n x^n = \sum_{n=0}^{N}\frac{d}{dx}[(-1)^n x^n] = \boldsymbol{\sum_{n=0}^{N}(-1)^n n x^{n-1}}$$

(Note that the coefficient $(-1)^n n$ is zero when $n = 0$, hence we can start the summation with $n = 1$)

$$= \sum_{n=1}^{N}(-1)^n n x^{n-1} = (-1)^1 1x^0 + (-1)^2 2x^1 + (-1)^3 3x^2 + \cdots + (-1)^N N x^{N-1}$$

(observe that this can be written as a finite series involving x^n)

$$= \boldsymbol{\sum_{n=0}^{N-1}(-1)^{n+1}(n+1)x^n}$$

(This is an example of applying the technique of index-shifting: we increase the running index by one in the summand – replace *n* by $n+1$, and decrease the value of the running index in the summation notation by one – *n* runs from 0 to $N-1$ instead.)

Exercise Evaluate and simplify the following derivatives.

- $\frac{d}{dx}\sum_{i=1}^{n}[y_i - a - x{t_i}^3]^2$ [Answer: $-2(\sum_{i=1}^{n}{t_i}^3 y_i - a\sum_{i=1}^{n}{t_i}^3 - x\sum_{i=1}^{n}{t_i}^6)$]
- For $y = \sum_{N=0}^{\infty} a_N x^N$, evaluate and simplify $x\frac{d^2y}{dx^2}$.

  [Answer: $\sum_{N=2}^{\infty} N(N-1)a_N x^{N-1} = \sum_{N=1}^{\infty}(N+1)N a_{N+1}x^N$]

Derivative of Trigonometric Functions

Basic Formula	Basic Formula with Chain Rule
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin u = \cos u\,\frac{du}{dx}$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos u = -\sin u\,\frac{du}{dx}$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan u = \sec^2 u\,\frac{du}{dx}$
$\frac{d}{dx}\csc x = -\csc x\cot x$	$\frac{d}{dx}\csc u = -\csc u\cot u\,\frac{du}{dx}$
$\frac{d}{dx}\sec x = \sec x\tan x$	$\frac{d}{dx}\sec u = \sec u\tan u\,\frac{du}{dx}$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\frac{d}{dx}\cot u = -\csc^2 u\,\frac{du}{dx}$

Exercise Evaluate and simplify the following derivatives.

- $\frac{d}{dt}\,(t - \sin t)$ [Answer: $1 - \cos t$]
- $\frac{d}{dt}\,(\sec t - 1)$ [Answer: $\sec t \tan t$]
- $\frac{d}{dt}\sin(2t)$ [Answer: $2\cos(2t)$]
- $\frac{d}{dr}\sin(3r^2)$ [Answer: $6r\cos(3r^2)$]

Derivative of Inverse Trigonometric Functions

Basic Formula	Basic Formula with Chain Rule
$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$	$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

Exercise Evaluate and simplify the following derivatives.

- $\frac{d}{dx}\,[\tan^3(2x) + \tan^{-1}(2x)]$ [Answer: $6\tan^2(2x)\sec^2(2x) + \frac{2}{1+4x^2}$]

Derivative of Exponential and Logarithmic Functions

Basic Formula	Basic Formula with Chain Rule
$\frac{d}{dx}\,b^x = b^x \ln b$	$\frac{d}{dx}\,b^u = b^u \ln b\,\frac{du}{dx}$
$\frac{d}{dx}\,e^x = e^x$	$\frac{d}{dx}\,e^u = e^u\,\frac{du}{dx}$
$\frac{d}{dx}\log_b x = \frac{1}{x}\log_b e$	$\frac{d}{dx}\log_b u = \frac{1}{u}\log_b e\,\frac{du}{dx}$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

Exercise Evaluate and simplify the following derivatives.

- $\frac{d}{dt}(a^2e^t + 5b\cos c + a^3\sin 3c - b^4t - 12)$ [Answer: $a^2e^t - b^4$]
- $\frac{d}{dt}\,[\cos(8t) + e^{2t}]$ [Answer: $-8\sin(8t) + 2e^{2t}$]
- $\frac{d}{dy}\,\ln(3y^4)$ [Answer: $\frac{4}{y}$]

- $\frac{d}{du}\left(\frac{2}{3}\ln\frac{5-u}{2}\right)$ [Answer: $\frac{2}{3(u-5)}$]
- $\frac{d}{dy}\ln(\cos e^{2y})$ [Answer: $-2e^{2y}\tan e^{2y}$]
- $\frac{d}{du}\ln\left[(\sin 2u)\sqrt{u^2+1}\right]$ [Answer: $2\cot 2u + \frac{u}{u^2+1}$]
- $\frac{d}{dt}\ln\frac{t-1}{t+1}$ [Answer: $\frac{2}{(t-1)(t+1)}$ or $\frac{2}{t^2-1}$]

<u>Derivative of a Power of a Function</u>

Basic Formula	Basic Formula with Chain Rule
$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$	$\frac{d}{dx}\left(u^n\right)=nu^{n-1}\cdot\frac{d}{dx}(u)$
$\frac{d}{dx}\left(x^{p/q}\right)=\frac{p}{q}x^{(p/q)-1}$	$\frac{d}{dx}\left(u^{p/q}\right)=\frac{p}{q}u^{(p/q)-1}\cdot\frac{d}{dx}(u)$

<u>Example</u> Evaluate $\frac{d}{dt}\ln^3(1-2t)$.

Solution:

$$\frac{d}{dt}\ln^3(1-2t)=\frac{d}{dt}\{[\ln(1-2t)]^3\}=3[\ln(1-2t)]^2\cdot\frac{d}{dt}[\ln(1-2t)]$$

$$=3[\ln(1-2t)]^2\cdot\frac{1}{1-2t}\cdot\frac{d}{dt}(1-2t)=3[\ln(1-2t)]^2\cdot\frac{1}{1-2t}\cdot(-2)=-\mathbf{\frac{6}{1-2t}\ln^2(1-2t)}$$

<u>Exercise</u> Evaluate and simplify the following derivatives.

- $\frac{d}{dx}\{2[3+\cot(4x)]^3\}$ [Answer: $-24[3+\cot(4x)]^2\csc^2(4x)$]
- $\frac{d}{dx}\sqrt{1+\cos(2x)}$ [Answer: $\frac{\sin(2x)}{\sqrt{1+\cos(2x)}}$]
- $\frac{d}{dt}[15\cos^2(120\pi t)]$ [Answer: $-3600\pi\sin(120\pi t)\cos(120\pi t)$]
- $\frac{d}{dx}\left[\left(e^{1/x}\right)^2\right]$ [Answer: $-\frac{2}{x^2}e^{2/x}$]
- $\frac{d}{du}[\sec(2u)+\tan(2u)]^3$ [Answer: $6\sec(2u)[\sec(2u)+\tan(2u)]^3$]
- $\frac{d}{dx}\sqrt{x^2+1}$ [Answer: $\frac{x}{\sqrt{x^2+1}}$]
- $\frac{d}{dx}\frac{1}{(1-4x)^5}$ [Answer: $\frac{20}{(1-4x)^6}$]

- The electric resistance R of a wire varies inversely as the square of its radius r. For a given wire, $R = 4.66\Omega$ for $r = 0.15\text{mm}$. Find the derivative of R with respect to r for this wire. [Answer: $\frac{dR}{dr} = -\frac{0.21}{r^3}$]

Derivative of Products and Quotients

$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

Example Evaluate $\frac{d}{dx}\,(3e^{4x} + 4x^2\ln x)^2$.

Solution:

(First apply the general power rule: power rule + chain rule)

$$\frac{d}{dx}\,(3e^{4x} + 4x^2\ln x)^2 = 2(3e^{4x} + 4x^2\ln x)\frac{d}{dx}(3e^{4x} + 4x^2\ln x)$$

(Apply the product rule for the derivative of the second term)

$$= 2(3e^{4x} + 4x^2\ln x)\cdot\left[3e^{4x}\cdot\frac{d}{dx}(4x) + (4x^2)\frac{d}{dx}\ln x + (\ln x)\frac{d}{dx}(4x^2)\right]$$

$$= 2(3e^{4x} + 4x^2\ln x)\cdot\left[3e^{4x}\cdot 4 + (4x^2)\cdot\frac{1}{x} + (\ln x)\cdot(8x)\right]$$

$$= 2(3e^{4x} + 4x^2\ln x)\cdot[12e^{4x} + 4x + 8x\ln x]$$

(Factor out common factor 4 from the three terms in the last factor)

$$= \mathbf{8(3e^{4x} + 4x^2\ln x)\cdot[3e^{4x} + x + 2x\ln x]}$$

Example Evaluate $\frac{d}{dt}\ln\left(\sqrt[3]{\frac{t}{t^3+1}}\right)$

Solution:

(Apply Chain Rule) $\left(\frac{d}{du}\ln u\Big|_{u=\sqrt[3]{\frac{t}{t^3+1}}}\right)\cdot\left(\frac{d}{du}\sqrt[3]{u}\Big|_{u=\frac{t}{t^3+1}}\right)\cdot\left(\frac{d}{dt}\frac{t}{t^3+1}\right)$

(Apply basic differentiation formulae for the first two and the quotient rule for the last)

$$= \left(\frac{1}{u}\Big|_{u=\sqrt[3]{\frac{t}{t^3+1}}}\right)\cdot\left(\frac{1}{3}u^{-2/3}\Big|_{u=\frac{t}{t^3+1}}\right)\cdot\frac{(t^3+1)(1)-(t)(3t^2)}{(t^3+1)^2}$$

$$= \left(\frac{t}{t^3+1}\right)^{-1/3}\cdot\frac{1}{3}\left(\frac{t}{t^3+1}\right)^{-2/3}\cdot\frac{t^3+1-3t^3}{(t^3+1)^2} = \frac{1}{3}\left(\frac{t}{t^3+1}\right)^{-1}\frac{1-2t^3}{(t^3+1)^2}$$

$$= \frac{(t^3+1)(1-2t^3)}{3t(t^3+1)^2} = \mathbf{\frac{1-2t^3}{3t(t^3+1)}}$$

Exercise Evaluate and simplify the following derivatives.

- $\frac{d}{dx}\left[2x^3(3-x^3)^4\right]$ [Answer: $6x^2(3-5x^3)(3-x^3)^3$]
- $\frac{d}{dx}\left[x^3\cos(x^2+1)\right]$ [Answer: $3x^2\cos(x^2+1)-2x^4\sin(x^2+1)$]
- $\frac{d}{dt}\left[t\csc^3(2t)\right]$ [Answer: $\csc^3(2t)\left[1-6t\cot(2t)\right]$]
- $\frac{d}{d\theta}\left[e^{-2\theta}\sin(3\theta)\right]$ [Answer: $e^{-2\theta}\left[3\cos(3\theta)-2\sin(3\theta)\right]$]
- $\frac{d}{dx}(e^{4x}\cos x)$ [Answer: $e^{4x}(4\cos x-\sin x)$]
- $\frac{d}{dt}\left(e^t\cos\frac{3t}{4}\right)$ [Answer: $e^t\left(\cos\frac{3t}{4}-\frac{3}{4}\sin\frac{3t}{4}\right)$]
- $\frac{d}{dx}\left[e^{x^3}\cos(2x)\right]$ [Answer: $e^{x^3}\left[3x^2\cos(2x)-2\sin(2x)\right]$]
- $\frac{d}{dx}\left(xe^{\tan x}\right)$ [Answer: $e^{\tan x}(1+x\sec^2 x)$]
- $\frac{d}{dt}\left[(t^2+1)\tan^{-1}t-t\right]$ [Answer: $2t\tan^{-1}t$]
- $\frac{d}{du}\left(u\sin^{-1}2u+\frac{1}{2}\sqrt{1-4u^2}\right)$ [Answer: $\sin^{-1}2u$]
- $\frac{d}{dx}\left[2x(5-3x)^4\right]$ [Answer: $10(1-3x)(5-3x)^3$]
- $\frac{d}{dy}\frac{2y}{1-\cot 3y}$ [Answer: $\frac{2-2\cot 3y-6y\csc^2 3y}{(1-\cot 3y)^2}$]
- Find the derivative of $y=\frac{3x^2+x}{1-4x}$ at $(2,-2)$. [Answer: $-\frac{5}{7}$]
- Evaluate the derivative of $y=\frac{x}{\sqrt{1-4x}}$ for $x=-2$. [Answer: $\frac{5}{27}$]

Differentiation of Implicit Functions

To find the derivative dy/dx when y is defined as an implicit function of x, we differentiate each term of the equation with respect to x, regarding y as a differentiable function of x.

Example Given $y^2+xy=\sin(x+y)$, find $\frac{dy}{dx}$.

Solution:

(Since we cannot use the given equation to solve for y, that is, isolate y by itself and express y in terms of x alone, we use implicit differentiation and differentiate both sides of the equation with respect to x, keeping in mind that y is an implicit function of x)

$\frac{d}{dx}[y^2 + xy] = \frac{d}{dx}\sin(x+y)$

(For the left hand side, we differentiate the first term by using the chain rule and the second term by applying the product rule; for the right hand side, we use the chain rule)

$2y\frac{dy}{dx} + \left(\frac{dx}{dx}\cdot y + x\cdot\frac{dy}{dx}\right) = \cos(x+y)\cdot\left(\frac{dx}{dx}+\frac{dy}{dx}\right)$

$\Longrightarrow 2y\frac{dy}{dx} + \left(1\cdot y + x\cdot\frac{dy}{dx}\right) = \cos(x+y)\cdot\left(1+\frac{dy}{dx}\right)$

$\Longrightarrow 2y\frac{dy}{dx} + \left(y + x\cdot\frac{dy}{dx}\right) = \cos(x+y)\cdot\left(1+\frac{dy}{dx}\right)$

(Expand the equation and rearrange the terms so that all the terms involving $\frac{dy}{dx}$ appear on the left hand side)

$2y\frac{dy}{dx} + y + x\frac{dy}{dx} = \cos(x+y) + \cos(x+y)\frac{dy}{dx}$

$\Rightarrow 2y\frac{dy}{dx} + x\frac{dy}{dx} - \cos(x+y)\frac{dy}{dx} = \cos(x+y) - y$

or $\quad y - \cos(x+y) = \cos(x+y)\frac{dy}{dx} - x\frac{dy}{dx} - 2y\frac{dy}{dx}$

(Factor and solve for $\frac{dy}{dx}$)

$\Rightarrow [2y + x - \cos(x+y)]\frac{dy}{dx} = \cos(x+y) - y$

$\Rightarrow \boldsymbol{\frac{dy}{dx} = \frac{\cos(x+y)-y}{2y+x-\cos(x+y)}}$

or $\quad y - \cos(x+y) = [\cos(x+y) - x - 2y]\frac{dy}{dx}$

$\Rightarrow \boldsymbol{\frac{dy}{dx} = \frac{y-\cos(x+y)}{\cos(x+y)-x-2y}}$

Exercise

- Given $x^2 + y^4 = \sin(x+y)$, find $\frac{dy}{dx}$.

  [Answer: $\frac{dy}{dx} = \frac{\cos(x+y)-2x}{4y^3-\cos(x+y)}$ or $\frac{dy}{dx} = \frac{2x-\cos(x+y)}{\cos(x+y)-4y^3}$]

- Given $\cot(2x) - 3\csc(xy) = y^2$, find $\frac{dy}{dx}$. [Answer: $\frac{dy}{dx} = \frac{2\csc^2(2x)-3y\csc(xy)\cot(xy)}{3x\csc(xy)\cot(xy)-2y}$]

- Find $\frac{dy}{dx}$ if $3y^4 + xy^2 + 2x^3 - 6 = 0$. [Answer:]

- Find $\frac{dy}{dx}$ if $2x^3y + (x+y^2)^3 = x^4$. [Answer: $\frac{dy}{dx} = \frac{4x^3-6x^2y-3(x+y^2)^2}{2x^3+6y(x+y^2)^2}$]

- Find $\frac{dy}{dx}$ if $(1+y^2)^3 - x^2y = 7x$. [Answer: $\frac{dy}{dx} = \frac{7+2xy}{6y(1+y^2)^2-x^2}$]

- Find the derivative $\frac{dy}{dx}$ if $y\sec(2x)=\sin^{-1}(3y)$. [Answer: $\frac{dy}{dx}=\frac{2y\sqrt{1-9y^2}\sec(2x)\tan(2x)}{3-\sqrt{1-9y^2}\sec(2x)}$]

Higher Derivatives

Example Find the second derivative of $y=\frac{2x}{3x+2}$.

Solution:

(Apply the quotient rule to find the first derivative)

$$\frac{dy}{dx}=\frac{(3x+2)\frac{d}{dx}(2x)-(2x)\frac{d}{dx}(3x+2)}{(3x+2)^2}=\frac{(3x+2)\frac{d}{dx}(2x)-(2x)\frac{d}{dx}(3x+2)}{(3x+2)^2}$$

$$=\frac{(3x+2)(2)-(2x)(3)}{(3x+2)^2}=\frac{6x+4-6x}{(3x+2)^2}=\frac{4}{(3x+2)^2}$$

(To find the second derivative, apply either the quotient rule one more time, or more simply the general power rule]

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{4}{(3x+2)^2}\right)=\frac{d}{dx}[4\cdot(3x+2)^{-2}]$$

$$=4(-2)(3x+2)^{-3}\frac{d}{dx}(3x+2)=4(-2)(3x+2)^{-3}(3)=-\frac{\mathbf{24}}{\mathbf{(3x+2)^3}}$$

Example Find y'' for the implicit function defined by $2x^2+3y^2=6$.

Solution:

(Use implicit differentiation: differentiate both sides with respect to *x*)

$$2(2x)+3(2y)\frac{dy}{dx}=0\Longrightarrow 4x+6y\frac{dy}{dx}=0\Longrightarrow 2x+3y\frac{dy}{dx}=0$$

(Use implicit differentiation the second time to find y'')

$$2(1)+3\left[\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)+y\frac{d^2y}{dx^2}\right]=0\Longrightarrow 2+3\left[\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right]=0$$

(Substitute the result for $\frac{dy}{dx}$ from the first equation into the second equation and solve for the second derivative) $2x+3y\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{2x}{3y}$

$$\Rightarrow 2+3\left[\left(-\frac{2x}{3y}\right)^2+y\frac{d^2y}{dx^2}\right]=0\Rightarrow 2+\frac{4x^2}{3y^2}+3y\frac{d^2y}{dx^2}=0\Rightarrow\frac{d^2y}{dx^2}=-\frac{1}{3y}\left(2+\frac{4x^2}{3y^2}\right)$$

(Simplify the result by using the original given equation)

$2x^2 + 3y^2 = 6 \Rightarrow 2x^2 = 6 - 3y^2$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{3y}\left[2 + \frac{2(2x^2)}{3y^2}\right] = -\frac{1}{3y}\left[2 + \frac{2(6-3y^2)}{3y^2}\right] = -\frac{1}{3y} \cdot \frac{6y^2+12-6y^2}{3y^2} = -\frac{12}{9y^3} = -\frac{\mathbf{4}}{\mathbf{3y^3}}$

Exercise

- For $f(x) = \sec x$, find $f'\left(\frac{\pi}{6}\right)$ and $f''\left(\frac{\pi}{6}\right)$. [Answer: $f'\left(\frac{\pi}{6}\right) = \frac{2}{3}$, $f''\left(\frac{\pi}{6}\right) = \frac{10}{9}\sqrt{3}$]
- Find the higher derivatives of $f(x) = x(x^2 - 1)^2$. [Answer: $f'(x) = 5x^4 - 6x^2 + 1$, $f''(x) = 20x^3 - 12x$, $f'''(x) = 60x^2 - 12$, $f^{(4)}(x) = 120x$, $f^{(5)}(x) = 120$, $f^{(n)}(x) = 0$ for $n \geq 6$]
- Evaluate the second derivative of $y = \frac{2}{1-x}$ for $x = -2$. [Answer: $\frac{4}{27}$]