Complex Numbers

Equality and Representations

- $i = \sqrt{-1}, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$
- Square Root of a Negative Number : For $b > 0, \sqrt{-b} = i\sqrt{b}$
- Equality of complex nos.: $a + bi = c + di \iff a = c$ and $b = d$
- Graphical Representation of a complex no.
- 3 algebraic forms:
  - Rectangular Form $x + yi$ [x is the real part, y is the imaginary part]
  - Polar Form $r(\cos\theta + i\sin\theta) = r\angle\theta$ [r is the modulus and θ is an argument in radians; θ is the principal argument if $0 \le \theta < 2\pi$]
  - Exponential Form $re^{i\theta}$
- Conversion of a complex no. in one form to another form
  - $x = r\cos\theta$, $y = r\sin\theta$
  - $r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$

Addition/Subtraction and scalar multiplication

- algebraically using rectangular form (collect real parts and collect imaginary parts)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

- graphically (parallelogram rule or tip-to-tail rule)

Example For $z_1 = 500 + 0i$ and $z_2 = 40\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$, find $|z_1 - z_2|$ (rounded to two decimal places) and $\arg(z_1 - z_2)$ (rounded to the nearest tenth of a degree).

Solution:

(We first convert z_2 to rectangular form, the preferred form for subtraction)

$$z_2 = 40\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 40\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -20\sqrt{2} - 20\sqrt{2}i$$

(Subtract z_2 from z_1)

$$z_1 - z_2 = (500 + 0i) — 20\sqrt{2} - 20\sqrt{2}i = \left(500 + 20\sqrt{2}\right) + 20\sqrt{2}i$$

(Compute $|z_1 - z_2|$) $|z_1 - z_2| = \sqrt{\left(500 + 20\sqrt{2}\right)^2 + \left(20\sqrt{2}\right)^2} =$

$$= \sqrt{500^2 + 2(500)\left(20\sqrt{2}\right) + \left(20\sqrt{2}\right)^2 + \left(20\sqrt{2}\right)^2} =$$

$$= \sqrt{250000 + 20000\sqrt{2} + 800 + 800} = \sqrt{251600 + 20000\sqrt{2}} \approx \mathbf{529.04}$$

(To find $\arg(z_1 - z_2)$, we first find the reference angle ϕ)

$$\phi = \tan^{-1}\left(\left|\frac{20\sqrt{2}}{500+20\sqrt{2}}\right|\right) = \tan^{-1}\left(\frac{\sqrt{2}}{25+\sqrt{2}}\right)$$

(Since both $20\sqrt{2} > 0$ and $500 + 20\sqrt{2} > 0$, $\theta = \arg(z_1 - z_2)$ is in the first quadrant and hence is given by $\theta = \phi$[remember to set your calculator in radian mode]):

$$\theta = \phi = \tan^{-1}\left(\frac{\sqrt{2}}{25+\sqrt{2}}\right) \approx \mathbf{0.05349\ rad}$$

Exercise Simplify and write your answer in the rectangular form $a + bi$.

- $(8 + 6i) + (3 + 2i)$ [Answer: $11 + 8i$]
- $(4 + 5i) - (6 - 3i)$ [Answer: $-2 + 8i$]

Multiplication using rectangular form

FOIL & simplify using $i^2 = -1$

$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i$$

Exercise Simplify and write your answer in the rectangular form $a + bi$.

- $(1 + 2i)(1 + 3i)$ [Answer: $-5 + 5i$]
- $(5 + 7i)(5 - 7i)$ [Answer: 74]
- $(3 - 7i)^2$ [Answer: $-40 - 42i$]
- i^{75} [Answer: $-i$]

Division using rectangular form

Multiply both numerator and denominator by complex conjugate of the denominator

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

Complex Numbers

Example Perform the division $\frac{25-50i}{7-24i}$ and write your answer in the form $a+bi$.

Solution:

(Multiply the numerator and the denominator by $7+24i$, the complex conjugate of the denominator) $\frac{25-50i}{7-24i} = \frac{(25-50i)(7+24i)}{(7-24i)(7+24i)}$

(Expand both the numerator and the denominator, simplify using $i^2 = -1$)

$$\frac{(25-50i)(7+24i)}{(7-24i)(7+24i)} = \frac{175+600i-350i-1200i^2}{49+168i-168i-576i^2} = \frac{175+600i-350i+1200}{49+168i-168i+576} = \frac{1375+250i}{625}$$

$$= \frac{1375}{625} + \frac{250}{625}i = \mathbf{\frac{11}{5} + \frac{2}{5}}\boldsymbol{i}$$

Exercise Simplify and write your answer in the rectangular form $a+bi$.

- $\frac{1+i}{1-i}$ [Answer: i]

Multiplication/Division using polar form (and exponential form)

- $(r_1\angle\theta_1)\,(r_2\angle\theta_2) = (r_1 \times r_2)\,\angle(\theta_1+\theta_2)$
- $\frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle(\theta_1-\theta_2)$