Applications of Integration

Rectilinear Motion

Velocity: $v = \int a(t)\, dt$; Displacement: $s = \int v(t)\, dt$

Example During the initial stage of launching a spacecraft vertically, the acceleration a (in m/s^2) of the spacecraft is $a = 6t^2$. Find the height s of the spacecraft after 6.0 s if $s = 12$ m for $t = 0.0$ s and $v = 16\ m/s$ for $t = 2.0$ s.

Solution:

(Integrate the formula for acceleration to get the formula for velocity)

$$v = \int 6t^2\, dt = 6\left(\frac{1}{3}t^3\right) + C_1 = 2t^3 + C_1$$

(Use the given information $v(2) = 16$ to find C_1)

$$16 = 2(2^3) + C_1 \Longrightarrow 16 = 16 + C_1 \Longrightarrow C_1 = 0 \Longrightarrow v = 2t^3$$

(Integrate the formula found for the velocity to find that for the displacement, or height, s)

$$s = \int 2t^3\, dt = 2\left(\frac{1}{4}t^4\right) + C_2 = \frac{1}{2}t^4 + C_2$$

(Use the information $s(0) = 12$ to find C_2)

$$12 = \frac{1}{2}(0^4) + C_2 \Rightarrow 12 = 0 + C_2 \Longrightarrow C_2 = 12$$

$$\Rightarrow s = \frac{1}{2}t^4 + 12$$

(Find the height after 6.0 s) $s(6) = \frac{1}{2}(6^4) + 12 = \mathbf{660}$ **(m)**

Exercise

- A ball is thrown vertically from the top of a building 24.5 m high and hits the ground 5.0 s later. What initial velocity was the ball given? Use $g = 9.8\ m/s^2$.
  [Answer: 19.6 m/s upward]
- The velocity v of an object as a function of the time t is $v = 60 - 4t$. Find the expression for the displacement s if $s = 10$ for $t = 0$. [Answer: $s = 10 + 60t - 2t^2$]

Areas

- Between a curve and the x-axis: $A = \int_a^b y\, dx = \int_a^b f(x)\, dx$
- Between a curve and the y-axis: $A = \int_c^d x\, dy = \int_c^d g(y)\, dy$
- Between two curves on x-axis: $A = \int_a^b (y_2 - y_1)\, dx$
- Between two curves on y-axis: $A = \int_c^d (x_2 - x_1)\, dy$

Example Find the area bounded by the parabola $y = x^2$ and the line $y = x + 2$.

Solution:

(Find intersection points of the curves)

$$x^2 = x + 2 \Longrightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1, 2$$

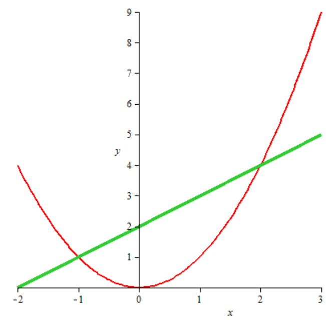
(Sketch the graphs)

$y = x^2$ is the standard parabola that opens up;

$y = x + 2$ is the straight line with slope $m = 1$

and

y-intercept $b = 2$



(Setup and evaluate an integral for the area)

$$\int_{-1}^{2}\left((x+2) - x^2\right) dx = \int_{-1}^{2}(x + 2 - x^2)\, dx = \left[\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right]_{-1}^{2}$$

$$= \left(\frac{1}{2}\cdot 2^2 + 2\cdot 2 - \frac{1}{3}\cdot 2^3\right) - \left(\frac{1}{2}\cdot(-1)^2 + 2\cdot(-1) - \frac{1}{3}\cdot(-1)^3\right)$$

$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = \mathbf{\frac{9}{2}}$$

Exercise

- Find the area in the first quadrant bounded by
  - $y = 9 - x^2$ [Answer: 18]
  - $y = \frac{1}{\sqrt{16 - x^2}}$ and $x = 3$ [Answer: $\sin^{-1}\frac{3}{4} \approx 0.8481$]
- Find the area bounded by
  - $y = x^2 + 1, y = 0, x = 0,$ and $x = 4$ [Answer: $\frac{76}{3}$]
  - $y = x^3, y = 0, x = 1,$ and $x = 2$ [Answer: $\frac{15}{4}$]
  - $y = 2x^2, y = 0, x = 1,$ and $x = 2$ [Answer: $\frac{14}{3}$]
  - $y = x^3 - 3$ and the lines $x = 2, y = -1,$ and $y = 3$

    [Answer: $8 - \frac{9}{2}\sqrt[3]{6} + \frac{3}{2}\sqrt[3]{2} \approx 1.713$]
  - $y = x^3 - 3x - 2$ and the x-axis [Answer: $\frac{27}{4}$]
  - $y = \frac{1}{4}x^2, y = 0$ and $x = 2$ [Answer: $\frac{2}{3}$]

- Find the first-quadrant area bounded by $y = \frac{e^{2x}}{\sqrt{e^{2x}+1}}$ and $x = 1.5$.

  [Answer: $\sqrt{e^3 + 1} - \sqrt{2} \approx 3.178$]

Volumes of Rotation

- Shell Method:

  $dV = 2\pi(\text{radius}) \times (\text{height}) \times (\text{thickness}) \Longrightarrow$

  $$\begin{cases} V = 2\pi \int_a^b x \cdot y\, dx = 2\pi \int_a^b x \cdot f(x)\, dx \\ V = 2\pi \int_c^d y \cdot x\, dy = 2\pi \int_c^d y \cdot g(y)\, dy \end{cases}$$

- Disk Method: $dV = \pi(\text{radius})^2 \times (\text{thickness}) \Longrightarrow \begin{cases} V = \pi \int_a^b y^2\, dx = \pi \int_a^b [f(x)]^2\, dx \\ V = \pi \int_c^d x^2\, dy = \pi \int_c^d [g(y)]^2\, dy \end{cases}$

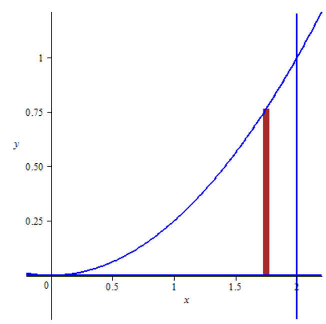
Example Find the volume of the solid generated by revolving the region bounded by $y = \frac{1}{4}x^2, y = 0$ and $x = 2$ about the x-axis.

Solution:

(Method 1: Rotate vertical elements about the x-axis – Disk Method)

$$V = \pi \int_a^b y^2\, dx = \pi \int_0^2 \left(\frac{1}{4}x^2\right)^2 dx = \pi \frac{1}{16} \int_0^2 x^4\, dx$$

$$= \frac{\pi}{16}\left(\frac{1}{5}x^5\right)\Big|_0^2 = \frac{\pi}{80}[2^5 - 0^5] = \frac{32}{80}\pi = \boldsymbol{\frac{2}{5}\pi}$$



(Method 2: Rotate horizontal elements about the x-axis – Shell Method)

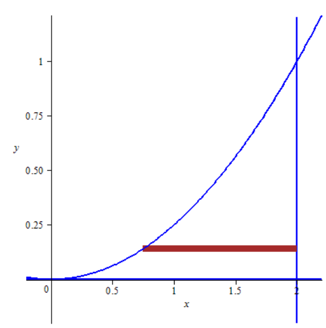
$y = \frac{1}{4}x^2 \Rightarrow x^2 = 4y \Longrightarrow x = +2\sqrt{y}$ (for the right half of the parabola)

$$V = 2\pi \int_c^d y \cdot x\, dy = 2\pi \int_0^1 y \cdot \left(2 - 2\sqrt{y}\right) dy$$

$$= 4\pi \int_0^1 \left(y - y\sqrt{y}\right) dy = 4\pi \int_0^1 \left(y - y^{3/2}\right) dy$$

$$= 4\pi \left[\frac{1}{2}y^2 - \frac{2}{5}y^{5/2}\right]\Big|_0^1 = 4\pi\left[\left(\frac{1}{2} - \frac{2}{5}\right) - (0 - 0)\right]$$

$$= 4\pi \cdot \frac{1}{10} = \frac{4}{10}\pi = \boldsymbol{\frac{2}{5}\pi}$$



Exercise

- Find the volume of the solid generated by revolving the first-quadrant region bounded by $y = 2x, y = 6$, and $x = 0$ about the y-axis. [Answer: 18π]
- Find the volume of the solid generated by revolving the region bounded by the curve $y = \frac{3}{\sqrt{4x+3}}, x = 2.5$, and the axes about the x-axis. [Answer: $\frac{9\pi}{4}\ln\frac{13}{3} \approx 10.4$]
- Find the volume of the solid generated by revolving the region bounded by $y = x^2, x = 0$ and $y = 9$ about the x-axis. [Answer: $\frac{972\pi}{5}$]

Centre of mass: $m_1d_1 + m_2d_2 + \cdots + m_nd_n = (m_1 + m_2 + \cdots + m_n)\bar{d}$

Centroid of an Area by Integration

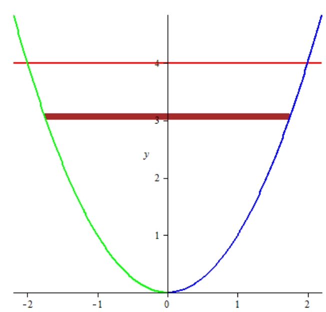
(a) If vertical elements are used, $\bar{x} = \dfrac{\int_a^b x(y_2 - y_1)\,dx}{\int_a^b (y_2 - y_1)\,dx}$

(b) If horizontal elements are used, $\bar{y} = \dfrac{\int_c^d y(x_2 - x_1)\,dy}{\int_c^d (x_2 - x_1)\,dy}$

Example Find the coordinates of the centroid of a thin plate covering the region bounded by the parabola $y = x^2$ and the line $y = 4$.

Solution:

$y = x^2 \Longrightarrow x_1 = -\sqrt{y}$ or $x_2 = +\sqrt{y}$



Using horizontal elements,

$$\bar{y} = \frac{\int_0^4 y[\sqrt{y}-(-\sqrt{y})]dy}{\int_0^4[\sqrt{y}-(-\sqrt{y})]dy} = \frac{\int_0^4 2y\sqrt{y}dy}{\int_0^4 2\sqrt{y}dy} = \frac{\cancel{2}\int_0^4 y^{3/2}dy}{\cancel{2}\int_0^4 y^{1/2}dy}$$

$$= \frac{(2/5)y^{5/2}\big|_0^4}{(2/3)y^{3/2}\big|_0^4} = \frac{(2/5)(4^{5/2}-0^{5/2})}{(2/3)(4^{3/2}-0^{3/2})} = \frac{(2/5)(32)}{(2/3)(8)} = \frac{12}{5}$$

Due to symmetry with respect to the vertical y-axis,, we expect $\bar{x} = 0$:

$$\bar{x} = \frac{\int_{-2}^2 x(4-x^2)dy}{\int_{-2}^2(4-x^2)dy} = \frac{\int_{-2}^2(4x-x^3)dy}{\int_{-2}^2(4-x^2)dy} = \frac{\left(2x^2-\frac{1}{4}x^4\right)\Big|_{-2}^2}{\left(4x-\frac{1}{3}x^3\right)\Big|_{-2}^2} = \frac{(8-4)-(8-4)}{\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)} = \frac{0}{32/3} = 0$$

Hence the centroid is $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) = \left(\mathbf{0}, \frac{\mathbf{12}}{\mathbf{5}}\right)$.

Exercise

- Find the centroid of an isosceles right triangle plate with side *a*.

  [Answer: The centroid is located at the point on the plate that is at a distance $a/3$ from each of the side of length *a*]

Radius of Gyration: $m_1 {d_1}^2 + m_2 {d_2}^2 + \cdots + m_n {d_n}^2 = (m_1 + m_2 + \cdots + m_n) R^2$

Moment of Inertia of Area: $I_y = k \int_a^b x^2 (y_2 - y_1)\, dx$; $I_x = k \int_c^d y^2 (x_2 - x_1)\, dy$

Example Find the moment of inertia and the radius of gyration of the plate (with uniform density *k*) covering the region bounded by $y = 4x^2, x = 1$, and the *x*-axis with respect to the *y*-axis.

Solution:

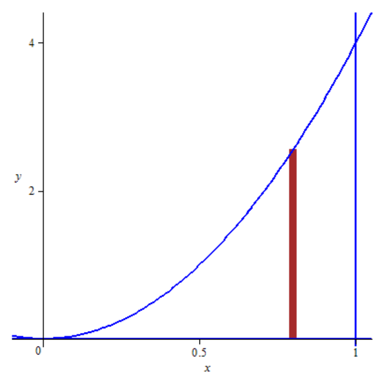
$$I_y = k \int_0^1 x^2 (y - 0)\, dx = k \int_0^1 x^2 (4x^2)\, dx$$

$$= 4k \int_0^1 x^4\, dx = 4k \left(\frac{1}{5}\right) x^5 \Big|_0^1 = \frac{4k}{5}(1 - 0) = \frac{\mathbf{4k}}{\mathbf{5}}$$

$$m = k \int_0^1 (y - 0)\, dx = k \int_0^1 (4x^2)\, dx$$

$$= 4k \int_0^1 x^2\, dx = 4k \left(\frac{1}{3}\right) x^3 \Big|_0^1 = \frac{4k}{3}(1 - 0) = \frac{4k}{3}$$

$$m{R_y}^2 = I_y \Rightarrow R_y = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{4k/5}{4k/3}} = \frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{\mathbf{15}}}{\mathbf{5}}$$



Exercise

- Find the moment of inertia of a right triangular plate with sides *a* and *b* with respect to the side *b*. Assume the plate the uniform density $k = 1$. [Answer: $\frac{1}{12} a^3 b$]
- Find the moment of inertial and the radius of gyration with respect to the *x*-axis of the solid (with uniform density *k*) generated by revolving the region bounded by the curves of $y^3 = x, y = 2$, and the *y*-axis about the *x*-axis. [Answer: $I_x = \frac{256\pi k}{7}, R_x = \frac{2}{7}\sqrt{35}$]
- Find the moment of inertia of a disk (of radius *r*) with respect to its axis and in terms of its mass. [Answer: $\frac{1}{2} m r^2$]

Example Find y in terms of x if $\frac{dy}{dx} = \sqrt{2x+1}$ and the curve $y = f(x)$ passes through the point (0,2).

Solution:

(Solve the differential equation using separation of variables; to integrate $\sqrt{2x+1}$, use the simple substitution $u = 2x + 1, du = 2\,dx$)

$$\frac{dy}{dx} = \sqrt{2x+1} \Longrightarrow dy = \sqrt{2x+1}\,dx \Longrightarrow \int dy = \int \sqrt{2x+1}\,dx$$

$$\Longrightarrow y = \frac{1}{2}\int \sqrt{2x+1}\,(2)dx = \frac{1}{2}\cdot\frac{1}{3/2}(2x+1)^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

(Determine C by using the condition that $y = 2$ when $x = 0$)

$$2 = \frac{1}{3}(2\times 0+1)^{3/2} + C \Longrightarrow 2 = \frac{1}{3} + C \Longrightarrow C = 2 - \frac{1}{3} = \frac{5}{3}$$

$$\Longrightarrow \boldsymbol{y = \frac{1}{3}(2x+1)^{3/2} + \frac{5}{3}}$$

Exercise

- Find the equation of the curve for which $\frac{dy}{dx} = \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}$ if the curve passes through $(0,1)$.

  [Answer: $y = 2e^{\sqrt{x+1}} + 1 - 2e$]
- Find the equation of the curve for which $\frac{dy}{dx} = 3x - 1$ if the curve passes through $(1,4)$.

  [Answer: $y = \frac{3}{2}x^2 - x + \frac{7}{2}$]
- Find y in terms of x if $\frac{dy}{dx} = (6-x)^4$ and the curve passes through $(5,2)$.

  [Answer: $y = -\frac{1}{5}(6-x)^5 + \frac{11}{5}$]
- Solve the differential equation:
  - $\frac{dx}{dt} = \frac{x}{t^2+4}$ [Answer: $\ln x = \frac{1}{2}\tan^{-1}\frac{t}{2} + C$, or $x = C_1 e^{\left(\frac{1}{2}\tan^{-1}\frac{t}{2}\right)}$]
  - $2e^{3x}\sin y\;dx + e^x\csc y\;dy = 0$ [Answer: $e^{2x} - \cot y = C$, or $y = \tan^{-1}\left(\frac{1}{e^{2x}-C}\right)$]
- Solve the differential equation subject to the given condition:
  - $(x^2+1)^2\,dy + 4x\,dx = 0;\;\; x = 1$ when $y = 2$ [Answer: $y = \frac{2}{x^2+1} + 1$]
  - $(xy+y)\frac{dy}{dx} = 2;\;\; y = 2$ when $x = 0$ [Answer: $y = 2\sqrt{1+\ln(x+1)}$]

Integrating Combinations

$d(xy) = xdy + ydx$	$d(x^2 + y^2) = 2(xdx + ydy)$
$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$	$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

Exercise Find the general solution of the differential equation $xdx + ydy = x^2dx + y^2dx$.

[Answer: $\ln(x^2 + y^2) = 2x + C$]

First Order Linear DE: $dy + P(x)y\,dx = Q(x)\,dx$: Integrating Factor $e^{\int P(x)dx}$ (7-step process)

1) Rewrite the linear DE into the above standard differential form.
2) Identify P and evaluate $\int P(x)\,dx$ (without the constant of integration).
3) Find the integrating factor (IF) defined by $e^{\int P(x)\,dx}$.
4) Multiply both sides of the DE with the IF:
$$e^{\int P(x)\,dx}\,dy + P(x)e^{\int P(x)\,dx}y\,dx = Q(x)e^{\int P(x)\,dx}\,dx$$
5) Rewrite the Left hand side of the previous result as a single differential:
$$d\left(e^{\int P(x)\,dx} \cdot y\right) = Q(x)e^{\int P(x)\,dx}\,dx$$
6) Integrate both sides of the previous result with respect to x and solve for y.
7) If possible, determine the constant of integration.

Example Find the solution of the initial-value problem:

$$x^2y' + 2xy = 1 \quad (x > 0); \qquad y(1) = 3$$

Solution:

(Step 1) $x^2\frac{dy}{dx} + 2xy = 1 \Rightarrow dy + \frac{2}{x}y\,dx = \frac{1}{x^2}dx$

(Step 2) $\int P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x| = 2\ln x \quad (\text{as } x > 0)$

(Step 3) Integrating Factor (IF) $= e^{2\ln x} = e^{\ln(x^2)} = x^2$

(Step 4) $x^2\left(dy + \frac{2}{x}y\,dx\right) = x^2\left(\frac{1}{x^2}dx\right) \Rightarrow x^2\,dy + 2xy\,dx = dx$

(Step 5) $d(x^2y) = dx$

(Step 6) $\int d(x^2y) = \int dx \Rightarrow x^2y = x + C \Longrightarrow y = \frac{x+C}{x^2}$

(Step 7) $y(1) = 3 \Rightarrow \frac{1+C}{1^2} = 3 \Rightarrow 1 + C = 3 \Rightarrow C = 2$

Hence $\boldsymbol{y = \frac{x+2}{x^2}}$ **or** $\boldsymbol{y = \frac{1}{x} + \frac{2}{x^2}}$.

Electric Current: $i = \frac{dq(t)}{dt}$; Electric Charge: $q = \int i(t)\, dt$

Voltage across a capacitor: $V_C = \frac{1}{C}\int i(t)\, dt$

Example A certain capacitor is measured to have a voltage of 100 V across it. At this instant a current as a function of time given by $i = 0.06\sqrt{t}$ is sent through the circuit. After 0.25 s, the voltage is measured to be 140 V. What is the capacitance of the capacitor?

Solution:

$$V_C = \frac{1}{C}\int i(t)\, dt = \frac{1}{C}\int 0.06\sqrt{t}\, dt = \frac{0.06}{C}\cdot\frac{1}{3/2}t^{3/2} + C_1 = \frac{1}{25C}t^{3/2} + C_1$$

$$V_C(0) = 100 \Rightarrow \frac{1}{25C}\cdot 0^{3/2} + C_1 = 100 \Longrightarrow C_1 = 100$$

Hence $V_C = \frac{1}{25C}t^{3/2} + 100$

$$V_C(0.25) = 140 \Rightarrow \frac{1}{25C}\cdot 0.25^{3/2} + 100 = 140 \Longrightarrow \frac{1}{200C} = 40$$

$$\Longrightarrow C = \frac{1}{200\times 40} = \mathbf{0.000125(F)} \text{ or } \mathbf{125\ (\mu F)}$$

Exercise

- The current in a certain electric circuit as a function of time is given by $i = 6t^2 + 4$. Find an expression for the amount of charge q that passes a point in the circuit as a function of time. Assuming that $q = 0$ when $t = 0$, determine the total charge that passes the point in 2 s. [Answer: $q = 2t^3 + 4t + q_0$, 24 C]
- The voltage across a 5.0-μF capacitor is zero. What is the voltage after 20 ms if a current of 75 mA charges the capacitor? [Answer: 300 V]

Second Order Homogeneous Linear DE: $a_0D^2y + a_1Dy + a_2y = 0$

Auxiliary Equation: $a_0m^2 + a_1m + a_2 = 0$

D>0: Distinct Roots:	$y = c_1e^{m_1x} + c_2e^{m_2x}$
D=0: Repeated Roots:	$y = e^{mx}(c_1 + c_2x)$
D<0: Complex Roots:	$y = e^{\alpha x}(c_1\sin(\beta x) + c_2\cos(\beta x))$

$a_0D^2y + a_1Dy + a_2y = b \Rightarrow y = y_c + y_p$

Example Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x$.

Solution:

(First, we get y_c by solving the associated homogeneous DE: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$)

Auxiliary equation: $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

Hence $y_c = e^{2x}(C_1 + C_2 x)$ (or $y_c = C_1 e^{2x} + C_2 x e^{2x}$)

(Next, we determine the form of y_p by using the non-homogeneous terms on the right hand side together with all their derivatives)

$3\boxed{x} \xrightarrow{D} 3 = 3 \cdot \boxed{1} \xrightarrow{D} 0 \Rightarrow y_p = A(x) + B(1) = Ax + B$

(Note: If the function forms we found in y_p appear in y_c, we have to multiply them with the lowest power of the variable x to eliminate the duplication.)

(We complete this solution by finding the values of the parameters A and B in y_p with the method of undetermined coefficients) $y_p = Ax + B \Rightarrow y_p' = A \Rightarrow y_p'' = 0$

$y_p'' - 4y_p' + 4y_p = 3x \Rightarrow 0 - 4(A) + 4(Ax + B) = 3x \Longrightarrow -4A + 4Ax + 4B = 3x$

$$\left.\begin{cases} \text{Constants:} & -4A + 4B = 0 \quad \Rightarrow \quad B = A \\ x: & 4A = 3 \quad \Rightarrow \quad A = 3/4 \end{cases}\right\} \Rightarrow B = \frac{3}{4}$$

Hence $y = y_c + y_p \Rightarrow \boldsymbol{y = e^{2x}(C_1 + C_2 x) + \frac{3}{4}x + \frac{3}{4}}$

Exercise

- Find the general solution of the differential equation $y'' + 2y' + 5y = 0$.

  [Answer: $y = e^{-x}[C_1 \sin(2x) + C_2 \cos(2x)]$]

- Find the general solution of the differential equation $2D^2y - Dy = 2\cos x$.

  [Answer: $y = C_1 + C_2 e^{x/2} - \frac{2}{5}\sin x - \frac{4}{5}\cos x$]

- A mass of 0.5 kg stretches a spring for which $k = 32$ N/m. With the weight attached, the spring is pulled 0.3 m longer than its equilibrium length and released. Find the equation of the resulting motion, assuming no damping. (The acceleration due to gravity is 9.8 m/s^2.) [Answer: $x = 0.3\cos(8t)$]

<u>*LRC* Electric Circuits</u>: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$

<u>Exercise</u> Find the equation for the current as a function of the time (in s) in a circuit containing a 2-H inductance, an 8-Ω resistor, and a 6-V battery in series, if $i = 0$ when $t = 0$.

[Answer: $i = \frac{3}{4}(1 - e^{-4t})$]