Applications of Differentiation

Derivative: Instantaneous Rate of Change

Example Find the slope of the tangent line to the curve of $y = \ln\frac{2x-1}{x^2+1}$ at a point where $x = 2$.

Solution:

(Slope of the curve is given by $\frac{dy}{dx}$; we can certainly differentiate directly $\ln\frac{2x-1}{x^2+1}$, however, an easier way is to rewrite the function using properties of logarithm before we differentiate) $y = \ln\frac{2x-1}{x^2+1} = \ln(2x-1) - \ln(x^2+1)$

$\Longrightarrow \frac{dy}{dx} = \frac{d}{dx}\ln(2x-1) - \frac{d}{dx}\ln(x^2+1) = \frac{2}{2x-1} - \frac{2x}{x^2+1}$

(Since we are only interested at the slope of the curve at $x = 2$, rather than simplifying the result, we can simply substitute in the value of x)

$\Longrightarrow \left.\frac{dy}{dx}\right|_{x=2} = \frac{2}{2(2)-1} - \frac{2(2)}{2^2+1} = \frac{2}{3} - \frac{4}{5} = \frac{10-12}{15} = -\mathbf{\frac{2}{15}}$

Exercise

- Find the slope of a line tangent to the curve of $y = 4x^7 - x^4$ at the point $(1,3)$.

  [Answer: 24]
- Find the slope of a line tangent to the curve of $y = 3x^2 - \frac{4}{x^2}$ at the point $(2,11)$.

  [Answer: 13]
- Find the slope of a line tangent to the curve of $y = \frac{3e^{2x}}{x^2+1}$ at a point where $x = 1.275$.

  [Answer: ≈ 15.05]
- Find the slope of the tangent line to the curve of $y = \frac{\tan^{-1}x}{x^2+1}$ at a point where $x = 3.60$.

  [Answer: ≈ –0.0429]
- Find the slope of a line tangent to the curve of $y = 5\sin(3x)$ at a point where $x = 0.2$.

  [Answer: $15\cos 0.6 \approx 12.38$]
- Find the slope of a line tangent to the curve of $2y^3 + xy + 1 = 0$ at the point $(-3,1)$.

  [Answer: $-\frac{1}{3}$]
- Find the expression for the time rate of change of electric current that is given by the equation $i = 8e^{-t}\sin(10t)$, where t is the time.

  [Answer: $\frac{di}{dt} = 8e^{-t}[10\cos(10t) - \sin(10t)]$]

- A spherical balloon is being inflated. Find the expression for the instantaneous rate of change of the volume with respect to the radius. Evaluate this rate of change for a radius of 2.00 m. [Answer: $\frac{dV}{dr} = 4\pi r^2$; $16\pi\ (\mathrm{m}^2) \approx 50.3\ (\mathrm{m}^2)$]
- The power *P* produced by an electric current *i* in a resistor caries directly as the square of the current. Given that 1.2 W of power are produced by a current of 0.50 A in a certain resistor, find an expression for the instantaneous rate of change of power with respect to current. Evaluate this rate of change for $i = 2.5$ A. [Answer: $P = 4.8i^2$; $24\ \mathrm{W/A}$]
- The stress *S* on a hollow tube is given by $S = \frac{16DT}{\pi(D^4 - d^4)}$ where *T* is the tension, *D* is the outer diameter, and *d* is the inner diameter of the tube. Find the expression for the instantaneous rate of change of *S* with respect to *D*, with the other values being constant. [Answer: $\frac{dS}{dD} = -\frac{16T(3D^4+d^4)}{\pi(D^4-d^4)^2}$]
- Under certain conditions, due to the presence of a charge *q*, the electric potential *V* along a line is given by $V = \frac{kq}{\sqrt{x^2+b^2}}$ where *k* is a constant and *b* is the minimum distance from the charge to the line. Find the expression for the instantaneous rate of change of *V* with respect to *x*. [Answer: $\frac{dV}{dx} = -\frac{kqx}{(x^2+b^2)^{3/2}}$]

Related Rates

Example Trash is being compacted into a cubical volume. The edge of the cube is decreasing at the rate of $0.10\ \mathrm{m/s}$. When an edge of the cube is 1.25 m, how fast is the volume changing?

Solution:

(Recall the formula for the volume *V* of a cube with side *x*) $V = x^3$

(Differentiate both sides with respect to time *t* – apply the Chain Rule of the right hand side) $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = 3x^2 \frac{dx}{dt}$

(Substitute in the data values, converted to the appropriate units if necessary; notice that $\frac{dx}{dt}$ is negative since *x* is decreasing with respect to time *t*)

$= 3 \times 1.25^2(\mathrm{m}^2) \times (-0.1)(\mathrm{m/s}) = -0.46875\ (\mathrm{m}^3/s)$

Hence **the volume is decreasing at the rate of $\mathbf{0.46875\ (m^3/s)}$**.

Example The force F of gravity of the earth on a spacecraft varies inversely as the square of the distance r of the spacecraft from the center of the earth. A particular spacecraft weighs 4500 N on the launch pad (F = 4500 N for r =6370 km). Find the rate at which F changes later as the spacecraft moves away from the earth at the rate of 12 km/s, where r = 8500 km.

Solution:

(Start with the form of the equation relating F and r) $F = \frac{k}{r^2}$

(Use the given information to find the constant of proportionality k)

$$4{,}500 = \frac{k}{6{,}370^2} \Longrightarrow k = 4{,}500 \times 6{,}370^2 = 182{,}596{,}050{,}000(\mathrm{N} \cdot \mathrm{km}^2)$$

$$\Longrightarrow F = \frac{182{,}596{,}050{,}000}{r^2}$$

(Use the Chain Rule to find $\frac{dF}{dt}$) $\frac{dF}{dt} = \frac{dF}{dr} \times \frac{dr}{dt} = \frac{(182{,}596{,}050{,}000)(-2)}{r^3} \times \frac{dr}{dt}$

(Substitute in the data values, converted to the appropriate units if necessary)

$$= \frac{(182{,}596{,}050{,}000)(-2)(\mathrm{N} \cdot \mathrm{km}^2)}{8500^3(\mathrm{km}^3)} \times 12\left(\frac{\mathrm{km}}{\mathrm{s}}\right) \approx -7.1(\mathrm{N/s})$$

Hence **the force is decreasing approximately at the rate of 7.1 N/s**.

Exercise

- A balloon leaves the ground 250 m from an observer and rises at the rate of 5.0 m/s. How fast is the angle of elevation of the balloon increasing after 8.0 s?

  [Answer: $\frac{25}{1282} \approx 0.020$ rad/s]

- The voltage of a certain thermocouple as a function of the temperature is given by $E = 2.8 + 0.006T^2$. If the temperature is increasing at the rate of 1.00 ºC/min, how fast is the voltage changing when $T = 100°\mathrm{C}$? [Answer: increasing at the rate 4 V/min]

- The distance q that an image is from a certain lens in terms of p, the distance of the object from the lens, is given by $q = \frac{10p}{p-10}$. If the object distance is increasing at the rate of 0.2 cm/s, how fast is the image distance changing when $p = 15.0$ cm?

  [Answer: decreasing at the rate of 0.8 cm/s]

- A spherical balloon is being blown up such that its volume increases at the constant rate of 2.00 m^3/min. Find rate at which the radius is increasing when it is 3.00 m.

  [Answer: $\frac{1}{18\pi} \approx 0.0177$ (m/min)]

- Two cruise ships leave Vancouver, British Columbia, at noon. Ship A travels west at 12.0 km/h (before turning towards Alaska), and ship B travels south at 16.0 km/h (toward Seattle). How fast are they separating at 2 P.M.? [Answer: 20.0 km/h]

Differential form of a function $y = f(x)$ is defined as $dy = f'(x)dx$.

Example The edge of a cube of gold was measured to be 3.850 cm. From this value, the volume was found. Later it was discovered that the value of the edge was 0.02 cm too small.

(a) By approximately how much was the volume in error?

(b) Estimate the relative error in the volume.

Solution:

(a) (Recall the formula for the volume V of a cube with side x) $V = x^3$

(Estimate the error in the calculate volume, ΔV, by the differential in the volume dV)

$$\Delta V \approx dV = \frac{dV}{dx} \times \Delta x = 3x^2 \times \Delta x$$

(Substitute in the data values, converted to the appropriate units if necessary)

$$= 3 \times 3.850^2(\text{cm}^2) \times (-0.02 \text{ cm}) = \mathbf{-0.88935\ (cm^3)}$$

(b) (Estimate relative error $\frac{\Delta V}{V}$ by $\frac{dV}{V}$) $\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{3x^2(\Delta x)}{x^3} = \frac{3(\Delta x)}{x} = \frac{3(-0.02 \text{ cm})}{3.850 \text{ cm}} \approx -0.0155$

Hence **the relative error is approximately –0.016 (or –1.6%)**.

Exercise

- Find the differential:
  - $y = 3x^5 - x$ [Answer: $dy = (15x^4 - 1)\, dx$]
  - $s = (2t^3 - 1)^4$ [Answer: $ds = 24t^2(2t^3 - 1)^3\, dt$]
  - $z = \frac{4u}{u^2+4}$ [Answer: $dz = -\frac{4(u^2-4)}{(u^2+4)^2}\, du$]
  - $y = \frac{\cos^2(3x+1)}{x}$ [Answer: $dy = -\frac{6x \sin(3x+1)\cos(3x+1) + \cos^2(3x+1)}{x^2} dx$]

Linearization: $L(x) = f(a) + f'(a)(x - a)$

Example Find an equation for the normal line to the graph of $y = 3x - 4x^2$ at the point corresponding to $x = 1$.

Solution:

(Differentiate the given function) $\frac{dy}{dx} = 3 - 8x$

(Evaluate the derivative at $x = 1$ for the slope of tangent line)

$\left.\frac{dy}{dx}\right|_{x=1} = 3 - 8(1) = -5$

(Take the negative reciprocal of the result to get the slope of the normal line)

$m_{\text{normal}} = -\frac{1}{-5} = \frac{1}{5}$

(Use the point slope form to find the equation of the normal line, simplify the answer)

$x = 1, y = 3(1) - 4(1)^2 = 3 - 4 = -1$

$y - (-1) = \frac{1}{5}(x - 1) \Rightarrow y + 1 = \frac{1}{5}x - \frac{1}{5} \Rightarrow \boldsymbol{y = \frac{1}{5}x - \frac{6}{5}}$

or $5y = 5\left(\frac{1}{5}x - \frac{6}{5}\right) \Longrightarrow 5y = x - 6 \Rightarrow 6 = x - 5y \Longrightarrow \boldsymbol{x - 5y = 6}$ or $\boldsymbol{y = \frac{1}{5}x - \frac{6}{5}}$

Exercise

- Find the equation of the line tangent to the parabola $y = x^2 - 1$ at the point $(-2,3)$.
  [Answer: $y = -4x - 5$]
- Find the equation of the line tangent to the curve $y = x^4 - 3x^2$ at the point $(2,4)$.
  [Answer: $y = 20x - 36$]
- Find the equation of the line normal to the hyperbola $y = \frac{2}{x}$ at the point $(2,1)$.
  [Answer: $y = 2x - 3$]
- Linearize the function $y = \sqrt{2x + 4}$ for $a = 6$. [Answer: $L(x) = \frac{1}{4}x + \frac{5}{2}$]
- Find the y-intercept of the line that is normal to the curve of $y = 2x - \frac{1}{3}x^3$ at the point with x-coordinate 3. [Answer: $\left(0, -\frac{24}{7}\right)$]
- Find the equation of the normal line to the parabola $4y = x^2$ with a slope of –1.
  [Answer: $y = -x + 3$]
- Use the linearization of the function $f(x) = \sqrt{2x + 1}$ at $x = 4$ to approximate $\sqrt{9.06}$.
  [Answer: 3.01]

Example Find an equation for the tangent line at point (5, 2) on the curve $x^2y = 50e^{4x-10y}$.

Solution:

(Differentiate implicitly, treating *y* as an implicit function of *x*; apply the product rule for the left hand side and the chain rule for the right hand side)

$$\frac{d}{dx}(x^2y) = \frac{d}{dx}(50e^{4x-10y}) \Rightarrow \frac{d}{dx}(x^2)\cdot y + x^2 \cdot \frac{dy}{dx} = 50e^{4x-10y}\cdot\frac{d}{dx}(4x-10y)$$

$$\Rightarrow 2xy + x^2\frac{dy}{dx} = 50e^{4x-10y}\left(4 - 10\frac{dy}{dx}\right)$$

(Evaluate at $x = 5, y = 2$ and $\frac{dy}{dx} = m$, the slope of the tangent line)

$$2(5)(2) + 5^2m = 50e^{4(5)-10(2)}(4-10m) \Rightarrow 20 + 25m = 50e^0(4-10m)$$

$$\Longrightarrow 20 + 25m = 200 - 500m \Rightarrow 525m = 180 \Rightarrow m = \frac{180}{525} = \frac{12}{35}$$

(Use the point slope form to find the equation of the tangent line, simplify the answer)

$$y - 2 = \frac{12}{35}(x-5) \Rightarrow 35(y-2) = 12(x-5) \Longrightarrow 35y - 70 = 12x - 60$$

$$\Rightarrow \mathbf{12x - 35y = -10}$$

Graph Sketching

Example For what positive values of *x* is the function $f(x) = \frac{1}{4x^2-12x+34}$ decreasing?

Solution:

(Check the domain of the function; a rational function is defined when the denominator is non-zero) $4x^2 - 12x + 34 = 0 \Rightarrow 2(2x^2 - 6x + 17) = 0$

(Discriminant $b^2 - 4ac = (-6)^2 - 4(2)(17) = 36 - 136 = -100 < 0$, hence the denominator is non-zero for any real number *x* and is either always positive or always negative; using the test point $x = 0$, we conclude $4x^2 - 12x + 34 > 0$ for all real number *x*)

($f(x)$ is decreasing when $f'(x) < 0$)

$$f'(x) = (-1)(4x^2 - 12x + 34)^{-2}\cdot\frac{d}{dx}(4x^2 - 12x + 34) = -\frac{8x-12}{(4x^2-12x+34)^2}$$

$$8x - 12 = 0 \Rightarrow x = \frac{12}{8} = \frac{3}{2}; \quad 4x^2 - 12x + 34 = 0 \Rightarrow \text{No solution}$$

Interval	$\left(-\infty,\frac{3}{2}\right)$	$\left(\frac{3}{2},\infty\right)$
Test Point (one possibility)	0	2
Inequality $-\frac{8x-12}{(4x^2-12x+34)^2}<0$ Satisfied?	$-\frac{(-)}{(+)}<0?$	$-\frac{(+)}{(+)}<0?$
Part of the Solution?	No	Yes

Hence $f(x)=\frac{1}{4x^2-12x+34}$ is decreasing for $\boldsymbol{x>\frac{3}{2}}$.

Example Sketch the graph of $y=\frac{4}{x^2}-x$ by finding the intercepts, the symmetry, the horizontal/oblique asymptote, the vertical asymptotes, the domain, the range, the values of x for which the function is increasing, decreasing, concave up, and concave down and by finding any maximum points, minimum points, and points of inflection.

Solution:

x-intercept(s): (set $y=0$)

$\frac{4}{x^2}-x=0 \Leftrightarrow \frac{4-x^3}{x^2}=0 \Leftrightarrow 4-x^3=0 \Leftrightarrow x^3=4 \Leftrightarrow x=\sqrt[3]{4}$

Hence **$\boldsymbol{x}$-intercept at $\left(\sqrt[3]{4},0\right)$**.

y-intercept: (set $x=0$) does not exist, since the domain of the function $y=\frac{4}{x^2}-x$ does not contain 0.

Symmetry:

- about x-axis? Replacing (x,y) by $(x,-y)$ in the equation leads to $-y=\frac{4}{x^2}-x \Leftrightarrow$ $y=-\frac{4}{x^2}+x$, which is different from the original equation, hence no!
- about y-axis? Replacing (x,y) by $(-x,y)$ in the equation leads to $y=\frac{4}{(-x)^2}-(-x) \Leftrightarrow y=\frac{4}{x^2}+x$, which is different from the original equation, hence no!
- about the origin? Replacing (x,y) by $(-x,-y)$ in the equation leads to $-y=\frac{4}{(-x)^2}-(-x) \Leftrightarrow y=-\frac{4}{x^2}-x$, which is different from the original equation, hence no!

Asymptote for $y = \frac{4}{x^2} - x \Leftrightarrow y = \frac{4-x^3}{x^2}$

- Vertical Asymptote: (set denominator equal to zero) $x^2 = 0 \Leftrightarrow \boldsymbol{x = 0}$ **(V. A.)**
- Since the function is a rational function (that is, quotient of two polynomials) with the degree of the numerator one higher than that of the denominator, we have **an oblique asymptote $\boldsymbol{y = -x}$** (as $\frac{4}{x^2} \longrightarrow 0$ as $x \longrightarrow \pm\infty$).

The function $y = \frac{4}{x^2} - x$ is defined everywhere except for $x = 0$, hence **the domain is the set of all real numbers except 0**.

Monotonicity: $y = \frac{4}{x^2} - x = 4x^{-2} - x$

$\Rightarrow \frac{dy}{dx} = 4(-2)x^{-3} - 1 = -\frac{8}{x^3} - 1 = -\frac{8+x^3}{x^3} = -\frac{(2+x)(4-2x+x^2)}{x^3}$

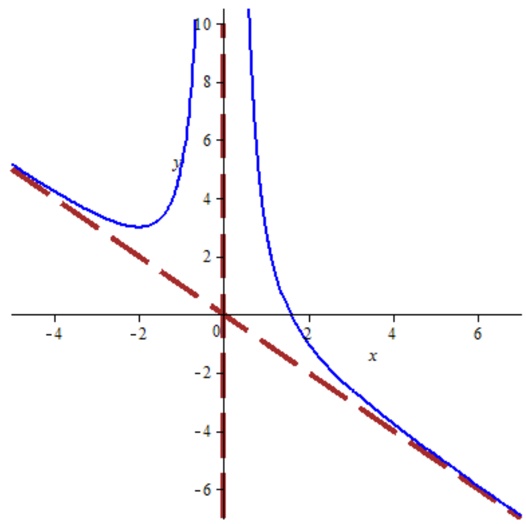
Interval	Test No. (an example)	Sign of $\frac{dy}{dx} = -\frac{(2+x)(4-2x+x^2)}{x^3}$	Function increasing or decreasing?
$(-\infty, -2)$	-3	$(-)\frac{(-)(+)}{(+)} = (+)$	Increasing
$(-2,0)$	-1	$(-)\frac{(+)(+)}{(+)} = (-)$	Decreasing
$(0, \infty)$	1	$(-)\frac{(+)(+)}{(+)} = (-)$	Decreasing

Hence **the function is decreasing for $\boldsymbol{x < -2}$ and $\boldsymbol{x > 0}$ and increasing for $\boldsymbol{-2 < x < 0}$**; moreover, **the function has a relative minimum at $\boldsymbol{(-2, 3)}$** $\left[x = -2, y = \frac{4}{(-2)^2} - (-2) = 1 + 2 = 3\right]$.

Concavity: $\frac{d^2y}{dx^2} = \frac{d}{dx}(-8x^{-3} - 1) = (-8)(-3)x^{-4} = \frac{24}{x^4} > 0$ when $x \neq 0$

Hence **the function is concave up for $\boldsymbol{x < 0}$ and $\boldsymbol{x > 0}$** and **there is no point of inflection**.

From the sketch below, we see that **the range is the set of all real numbers**.



Exercise Sketch the graph of the function

- $y = 6x - x^2$ [Answer: x-intercepts = (0,0), 6, y-intercept = (0,0); no symmetry; decreasing for $x > 3$, increasing for $x < 3$, relative maximum at (3,9); always concave down]
- $y = 2x^3 + 3x^2 - 12x$ [Answer: x-intercepts = $(0,0), \left(\frac{-3\pm\sqrt{105}}{4}, 0\right) \approx (-3.31,0), (0,0), (1.81,0)$, y-intercept = (0,0); no symmetry; decreasing on $-2 < x < 1$, increasing for $x < -2$ and $x > 1$, relative minimum at (1,–7), relative maximum at (–2, 20); concave down for $x < -\frac{1}{2}$, concave up for $x > -\frac{1}{2}$, point of inflection at $\left(-\frac{1}{2}, \frac{13}{2}\right)$]
- $y = x^5 - 5x^4$ [Answer: x-intercepts = (0,0), (5,0), y-intercept = (0,0); no symmetry; decreasing on $0 < x < 4$, increasing for $x < 0$ and $x > 4$, relative minimum at (4,–256), relative maximum at (0,0); concave down for $x < 3$,concave up for $x > 3$, point of inflection at (3,–162)]
- $y = \frac{8}{x^2+4}$ [Answer: no x-intercept, y-intercept = (0,2); symmetric to the y-axis; no V.A., H.A.: $y = 0$; decreasing for $x > 0$, increasing for $x < 0$, relative maximum at (0,2); concave down for $-\frac{2}{3}\sqrt{3} < x < \frac{2}{3}\sqrt{3}$, concave up for $x < -\frac{2}{3}\sqrt{3}$ and $x > \frac{2}{3}\sqrt{3}$, point of inflection at $\left(-\frac{2}{3}\sqrt{3}, \frac{3}{2}\right)$ and $\left(\frac{2}{3}\sqrt{3}, \frac{3}{2}\right)$]

- $y = x + \frac{4}{x}$ [Answer: no *x*- or *y*- intercepts; symmetric to the origin; V.A.: $x = 0$, oblique asymptote $y = x$; decreasing for $-2 < x < 0$ and $0 < x < 2$, increasing for $x < -2$ and $x > 2$, relative maximum at (–2,–4), relative minimum at (2,4); concave down for $x < 0$, concave up for $x > 0$]
- $y = \frac{1}{\sqrt{1-x^2}}$ [Answer: no *x*-intercept, *y*-intercept = $(0,1)$; symmetric to the *y*-axis; no H.A., V.A.: $x = -1, x = 1$; decreasing for $-1 < x < 0$ and increasing for $0 < x < 1$, relative minimum at (0,1); always concave up (for $-1, x < 1$)]
- $y = \frac{x}{x^2-4}$ [Answer: *x*-intercept = $(0,0)$, *y*-intercept = $(0,0)$; symmetric to the origin; H.A.: $y = 0$, V.A.: $x = -2, x = 2$; decreasing for all values in the domain; concave down for $x < -2$ and $0 < x < 2$, concave up for $-2 < x < 0$ and $x > 2$, point of inflection at (0,0)]

Exercise

- Sketch the graph of $y = x^3 + 6x^2$ by finding the values of *x* for which the function is increasing, decreasing, concave up, and concave down and by finding any maximum points, minimum points, and points of inflection. [Answer: decreasing for $-4 < x < 0$, increasing for $x < -4$ and $x > 0$, relative maximum at $(-4,32)$, relative minimum at $(0,0)$; concave down for $x < -2$, concave up for $x > -2$, point of inflection at $(-2,16)$]

Optimization Problem

Example Find the dimension of a 700-m^3 cylindrical storage tank that can be made with the least cost of sheet metal, assuming that there is no wasted sheet metal.

Solution:

(Recall the formula for the volume of a cylinder with radius *r* and height *h* is $V = \pi r^2 h$; substitute in the constraint to find a relationship between *r* and *h* that we can use to solve for one of them in terms of the other) $\pi r^2 h = 700 \Leftrightarrow h = \frac{700}{\pi r^2}$

(Use the result above to express the area of the cylinder, which represents the amount of sheet metal needed, in terms of only one variable)

$$A = 2\pi r^2 + 2\pi r h \Longrightarrow A = 2\pi r^2 + 2\pi r\left(\frac{700}{\pi r^2}\right) \Longrightarrow A = 2\pi r^2 + \frac{1400}{r}$$

(Find the candidate for the variable that minimizes the function by finding the critical numbers – the numbers at which the first derivative either vanishes or does not exist)

$$\frac{dA}{dr} = \frac{d}{dr}[2\pi r^2 + 1400r^{-1}] = (2\pi)(2r) + 1400(-1)r^{-2} = 4\pi r - \frac{1400}{r^2} = \frac{4\pi r^3 - 1400}{r^2}$$

Critical Numbers: $4\pi r^3 = 1400$ or $r^2 = 0 \quad \Longrightarrow r = \sqrt[3]{\frac{350}{\pi}}$ or $r = 0$

(Since $r = 0$, which leads to volume $V = \pi \times 0^2 \times h = 0$, obviously is not the answer, the remaining candidate must be the answer to this problem.)

$$\textbf{Radius } \boldsymbol{r} = \sqrt[3]{\frac{350}{\pi}} \approx \mathbf{4.81\ (m)} \Longrightarrow \textbf{Height } \boldsymbol{h} = \frac{700}{\pi\left(\sqrt[3]{\frac{350}{\pi}}\right)^2} = 2\left(\sqrt[3]{\frac{350}{\pi}}\right) \approx \mathbf{9.62\ (m)}$$

We can also double check our answer by showing the graph of the function is concave up

at $r = \sqrt[3]{\frac{350}{\pi}}$: $\quad \frac{d^2A}{dr^2} = \frac{d}{dr}\left[4\pi r - \frac{1400}{r^2}\right] = 4\pi - 1400(-2)r^{-3} = 4\pi + \frac{2800}{r^3}$

$$\Longrightarrow \left.\frac{d^2A}{dr^2}\right|_{r=\sqrt[3]{\frac{350}{\pi}}} = 4\pi + \frac{2800}{\left(\sqrt[3]{\frac{350}{\pi}}\right)^3} = 4\pi + \frac{2800}{350/\pi} = 4\pi + 8\pi = 12\pi > 0 \Longrightarrow \text{Concave Up}$$

Exercise

- Find the number that exceeds its square by the greatest amount. [Answer: $\frac{1}{2}$]
- An automobile manufacturer, in testing a new engine on one of its new models, found that the efficiency *e* of the engine as a function of the speed *s* of the car was given by $e = 0.768s - 0.00004s^3$ (here *e* is measured in percent and *s* is measured in km/h). What is the maximum efficiency of the engine?
  [Answer: maximum efficiency = 40.96% occurs when speed *s* = 80.0 km/h]
- A rectangular corral is to be enclosed with 1600 m of fencing. Find the maximum possible area of the corral. [Answer: 160000 m^2]
- The strength *S* of a beam with a rectangular cross section is directly proportional to the product of its width *w* and the square of its depth *d*. Find the dimensions of the strongest beam that can be cut from a log with a circular cross section that is 16.0 cm in diameter. [Answer: width = $\frac{16\sqrt{3}}{3}$ cm ≈ 9.24 cm, depth = $\frac{16}{3}\sqrt{6}$ cm ≈ 13.1 cm]
- Find the point on the parabola $y = x^2$ that is nearest to the point $(6,3)$. [Answer: $(2,4)$]

- The illuminance of a light source at any point equals the strength of the source divided by the square of the distance from the source. Two sources, of strengths 8 units and 1 unit, respectively, are 100 m apart. Determine at what point between them the illuminance is the least, assuming that the illuminance at any point is the sum of the illuminances of the two sources. [Answer: 66.7 m from the 8-unit source]
- The electric power (in W) produced by a certain source is given by $P = \frac{144r}{(r+0.6)^2}$, where r is the resistance (in Ω) in the circuit. For what value of r is the power a maximum? [Answer: 0.6]

Differential Equation

Exercise

- Verify that $y = e^{3x}$ is a solution of the differential equation $xy'' - y' - 3xy' + 3y = 0$.
- Verify that $y = cx + x^2$ is a solution of the differential equation $xy' = x^2 + y$.

Rectilinear Motion

$$\text{Average Velocity} = \frac{\Delta s}{\Delta t}; \text{Instantaneous Velocity } v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}\left(= \frac{ds}{dt}\right)$$

$$\text{Average Acceleration} = \frac{\Delta v}{\Delta t}; \text{Instantaneous Acceleration } a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}\left(= \frac{dv}{dt}\right)$$

Example The displacement s (in cm) of a pumping machine piston in each cycle is given by $s = t\sqrt{10 - 2t}$, where t is the time (in s). Find the velocity of the piston for $t = 4.00$ s.

Solution:

Velocity $v = \frac{ds}{dt} = \frac{d}{dt}\left[t\sqrt{10-2t}\right] = \frac{d}{dt}\left[t \cdot (10-2t)^{1/2}\right]$

(Apply Product Rule) $= 1 \cdot (10-2t)^{1/2} + t \cdot \frac{1}{2}(10-2t)^{-1/2}(-2)$

$= \sqrt{10-2t} - \frac{t}{\sqrt{10-2t}} = \frac{10-2t-t}{\sqrt{10-2t}} = \frac{10-3t}{\sqrt{10-2t}}$

$v(4) = \frac{10-3(4)}{\sqrt{10-2(4)}} = \frac{-2}{\sqrt{2}} = -\sqrt{2} \approx \mathbf{1.41\ (cm/s)}$

Exercise

- Find the expression for the instantaneous velocity of a falling object for which the distance s (in cm) fallen is given by $s = 490t^2$ where t is the time (in s). Determine the instantaneous velocity for $t = 3$ s. [Answer: $\frac{ds}{dt} = 980t;\ 2940\ \text{cm/s}$]
- For each 4.0-s cycle, the displacement s (in cm) of a piston is given by the equation $s = t^3 - 6t^2 + 8t$, where t is the time. Find the instantaneous velocity of the piston $\frac{ds}{dt}$ for $t = 2.6$s. [Answer: $-2.9\ \text{cm}/s$]
- A rocket is moving such that the only force on it is due to gravity and its mass is decreasing at a constant rate r. If it moves vertically, its velocity v as a function of the time t is given by $v = v_0 - gt - k\ln\left(1 - \frac{rt}{m_0}\right)$ where v_0 is the initial velocity, g is the acceleration due to gravity, t is the time, m_0 is the initial mass, and k is a constant. Determine an expression for the acceleration. [Answer: $\frac{dv}{dt} = \frac{kr}{m_0 - rt} - g$]

Curvilinear Motion

	Components	Magnitude	Direction
Velocity	$v_x = \frac{dx}{dt}$ and $v_y = \frac{dy}{dt}$	$v = \sqrt{{v_x}^2 + {v_y}^2}$	$\tan\theta_v = \frac{v_y}{v_x}$
Acceleration	$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$ and $a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$	$a = \sqrt{{a_x}^2 + {a_y}^2}$	$\tan\theta_a = \frac{a_y}{a_x}$

Example If the x- and y-coordinates of a moving object as functions of time are given by the parametric equations $x = 3t^2, y = 2t^3 - t^2$, find the magnitude and direction of the acceleration when $t = 2$.

Solution:

$$x = 3t^2 \Rightarrow \frac{dx}{dt} = 6t \Rightarrow a_x = \frac{d^2x}{dt^2} = 6 \Rightarrow a_x|_{t=2} = 6$$

$$y = 2t^3 - t^2 \Rightarrow \frac{dy}{dt} = 6t^2 - 2t \Rightarrow a_y = \frac{d^2y}{dt^2} = 12t - 2 \Rightarrow a_y\big|_{t=2} = 12(2) - 2 = 22$$

Magnitude: $a = \sqrt{6^2 + 22^2} = \sqrt{520} = \sqrt{4 \times 130} = \mathbf{2\sqrt{130} \approx 22.8}$

Direction: $\begin{Bmatrix} \tan\theta_a = \frac{a_y}{a_x} = \frac{22}{6} = \frac{11}{3} \\ a_x > 0 \text{ and } a_y > 0 \Longrightarrow 0 < \theta_a < \frac{\pi}{2} \end{Bmatrix}$

$\Longrightarrow \theta_a = \tan^{-1}\frac{11}{3} \approx \mathbf{74.7°}$ (**or 1.30 radians**)

Exercise

- If the horizontal distance *x* that an object has moved is given by $x = 3t^2$ and the vertical distance *y* is given by $y = 1 - t^2$, find the resultant velocity when $t = 2$.

  [Answer: $v = 4\sqrt{10} \approx 12.6, \theta_v = \tan^{-1}\left(-\frac{1}{3}\right) \approx -18.4°$]

- Find the velocity and direction of motion when $t = 2$ of an object moving such that its *x*- and *y*-coordinates of position are given by $x = 1 + 2t$ and $y = t^2 - 3t$.

  [Answer: $v = \sqrt{5} \approx 2.24, \theta_v = \tan^{-1}\frac{1}{2} \approx 26.6°$]

- In a physics experiment, a small sphere is constrained to move along parabolic path described by $y = \frac{1}{3}x^2$. If the horizontal velocity v_x is constant at 6.00 cm/s, find the velocity at the point (2.00, 1.33). [Answer: $v = \frac{10.0 \text{ cm}}{\text{s}}, \theta = \tan^{-1}\frac{4}{3} \approx 53.1°$]

- A rocket is taking off vertically at a distance of 6500 m from an observer. If, when the angle of elevation is 38.4°, it is changing at the rate of $5.0\ °/s$, how fast is the rocket ascending? [Answer: $924\ m/s$]

- A particle is moving so that its *x*- and *y*-coordinates are given by $x = \cos 2t$ and $y = \sin 2t$. Find the magnitude and direction of its velocity when $t = \frac{\pi}{8}$.

  [Answer: Magnitude = 2, $\theta = \frac{3\pi}{4}$]